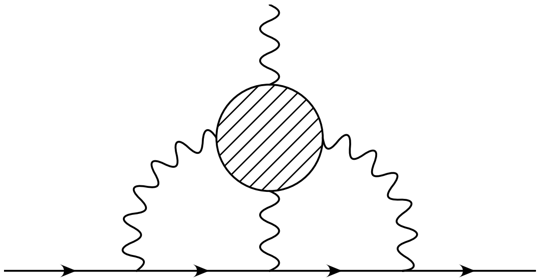
Muon g-2 Hadronic Light by Light : Latice QCD

Taku Izubuchi
(RBC&UKQCD)



BROOKHAVEN NATIONAL LABORATORY

RIKEN BNL Research Center

Symposium on Effective Field Theories and Lattice Gauge Theory,
May 18, 2016, Munich, Germany

Collaborators

- HLbL

Tom Blum, Norman Christ, Masashi Hayakawa, **<u>Luchang Jin</u>**, Chulwoo Jung, Christoph Lehner, ...

- DWF simulations including HVP

RBC/UKQCD Collaboration

Part of related calculation are done by resources from
USQCD (DOE), XSEDE, ANL BG/Q Mira (DOE, ALCC), Edinburgh BG/Q,
BNL BG/Q, RIKEN BG/Q and Cluster (RICC, HOKUSAI)

Support from US DOE, RIKEN, BNL, and JSPS

$(g-2)_\mu$ SM Theory prediction

- QED, EW, Hadronic contributions

K. Hagiwara et al. , J. Phys. G: Nucl. Part. Phys. 38 (2011) 085003

$$a_\mu^{\rm SM} = (11\ 659\ 182.8 \pm 4.9) \times 10^{-10}$$

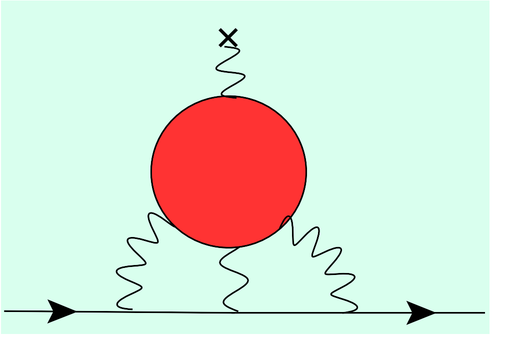
$$
\begin{aligned}
a_\mu^{\rm QED} &= (11\ 658\ 471.808 \pm 0.015) \times 10^{-10}\\
a_\mu^{\rm EW} &= (15.4 \pm 0.2) \times 10^{-10}\\
a_\mu^{\rm had,LOVP} &= (694.91 \pm 4.27) \times 10^{-10}\\
a_\mu^{\rm had,HOVP} &= (-9.84 \pm 0.07) \times 10^{-10}\\
a_\mu^{\rm had,lbl} &= (10.5 \pm 2.6) \times 10^{-10}
\end{aligned}
$$

[C. Lehner's talk]

$$\boxed{a_\mu^{\rm exp} - a_\mu^{\rm SM} = 28.8(6.3)_{\rm exp}(4.9)_{\rm SM} \times 10^{-10} \quad [3.6\sigma]}$$

- Discrepancy between EXP and SM is larger than EW!
- Currently the dominant uncertainty comes from HVP, followed by HLbL
- x4 or more accurate experiment FNAL , J-PARC
- Goal : sub 1% accuracy for HVP, and
  → **10% accuracy for HLbL**

Hadronic Light-by-Light



- 4pt function of EM currents
- No experimental data directly help
- Dispersive approach [Peter Stoffer's talk]

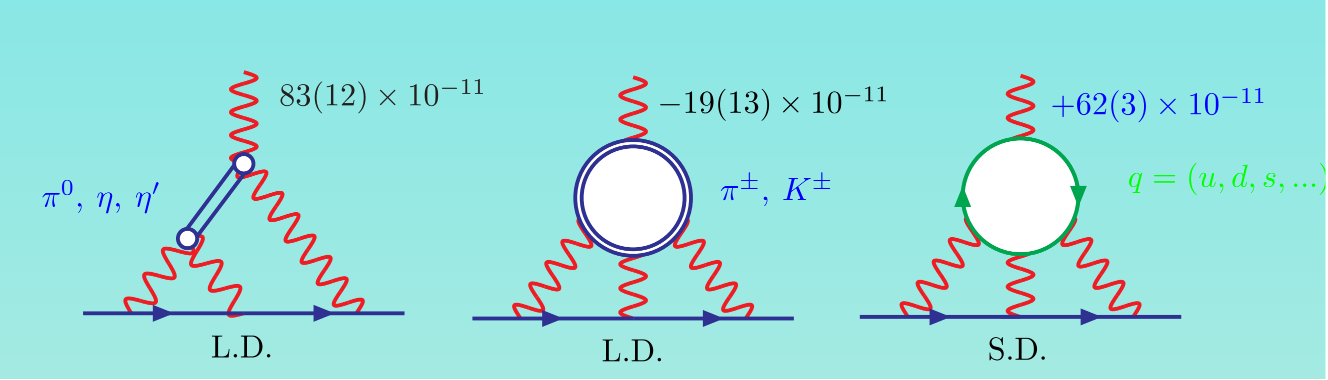
$$\Gamma_\mu^{(\mathrm{Hlbl})}(p_2, p_1) = ie^6 \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{\Pi^{(4)}_{\mu\nu\rho\sigma}(q, k_1, k_3, k_2)}{k_1^2\, k_2^2\, k_3^2} \times \gamma_\nu S^{(\mu)}(\not{p}_2 + \not{k}_2)\gamma_\rho S^{(\mu)}(\not{p}_1 + \not{k}_1)\gamma_\sigma$$

$$\Pi^{(4)}_{\mu\nu\rho\sigma}(q, k_1, k_3, k_2) = \int d^4x_1\, d^4x_2\, d^4x_3\, \exp[-i(k_1\cdot x_1 + k_2\cdot x_2 + k_3\cdot x_3)] \times \langle 0|T[j_\mu(0) j_\nu(x_1) j_\rho(x_2) j_\sigma(x_3)]|0\rangle$$

$$\text{Form factor}: \ \Gamma_\mu(q) = \gamma_\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2\, m_l} F_2(q^2)$$

HLbL from Models

- Model estimate with non-perturbative constraints at the chiral / low energy limits using anomaly : (9—12) x 10^{-10} with 25-40% uncertainty



F. Jegerlehner

Contribution	BPP	HKS	KN	MV	PdRV	N/JN
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	114±13	99±16
π, K loops	−19±13	−4.5±8.1	–	0±10	−19±19	−19±13
axial vectors	2.5±1.0	1.7±1.7	–	22± 5	15±10	22± 5
scalars	−6.8±2.0	–	–	–	−7± 7	−7± 2
quark loops	21± 3	9.7±11.1	–	–	2.3	21± 3
total	83±32	89.6±15.4	80±40	136±25	105±26	116±39

Direct 4pt calculation for selected kinematical range

- Jeremy Green arXiv: 1507.01577
- Compute connected contribution of 4 pt function in momentum space
- forward amplitudes related to $\gamma^*\gamma^*$-> hadron cross section via dispersion relation

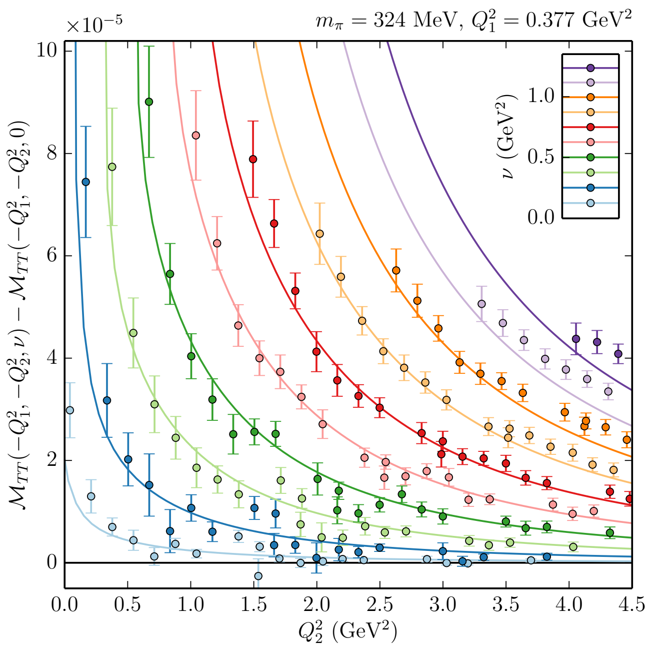


FIG. 3. The forward scattering amplitude $\mathcal{M}_{\mathrm{TT}}$ at a fixed virtuality $Q_1^2 = 0.377\mathrm{GeV}^2$, as a function of the other photon virtuality Q_2^2, for different values of ν. The curves represent the predictions based on Eq. (10), see the text for details.

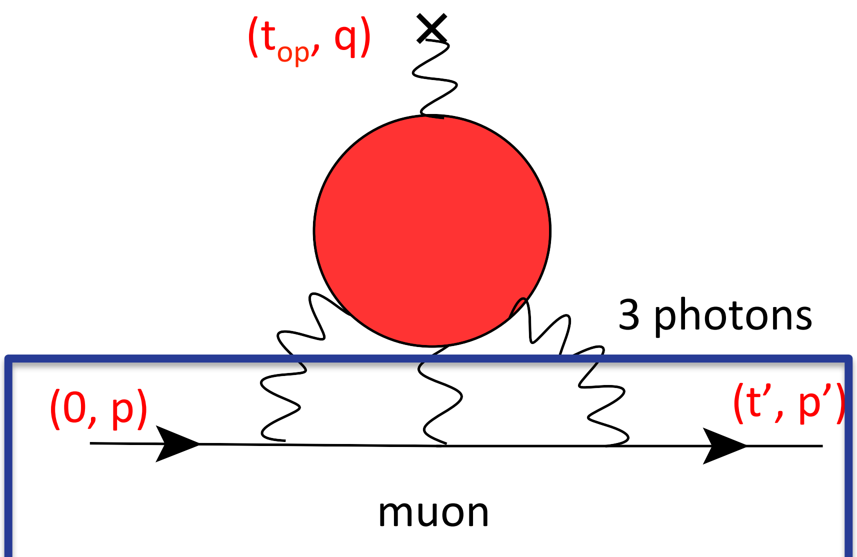
$$\nu \equiv q_1 \cdot q_2.$$

[Panel discussion]

Our Basic strategy : Lattice QCD+QED system [G. Schierholz's talk]

- 4pt function has too much information to parameterize (?)
- Do Monte Carlo integration for QED two-loop with 4 pt function $\pi^{(4)}$ which is sampled in lattice QCD
- Photon & lepton part of diagram is derived either in lattice QED+QCD [Blum et al 2014] (stat noise from QED), or exactly derive for given loop momenta [L. Jin et al 2015] (no noise from QED+lepton).

$$\Gamma_\mu^{(\mathrm{Hlbl})}(p_2,p_1) = ie^6 \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\Pi^{(4)}_{\mu\nu\rho\sigma}(q,k_1,k_2,k_3)$$
$$\times\left[S(p_2)\gamma_\nu S(p_2+k_2)\gamma_\rho S(p_1+k_1)\gamma_\sigma S(p_1) + (\mathrm{perm.})\right]$$

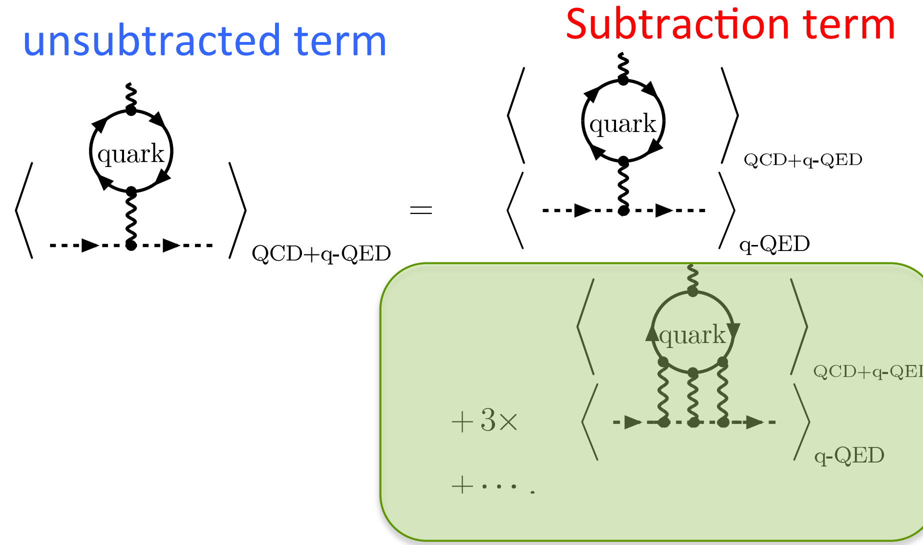


- set spacial momentum for
  - external EM vertex q
  - in- and out- muon p, p'

  q = p-p'
- set time slice of muon source(t=0), sink(t') and operator (t_{op})
- take large time separation for ground state matrix element

QCD+QED method [Blum et al 2015]

- One photon is treated analytically
- other two sampled stochastically
- needs subtraction
- use AMA for error reduction
- use Furry's theoretm to reduce α^2 noise

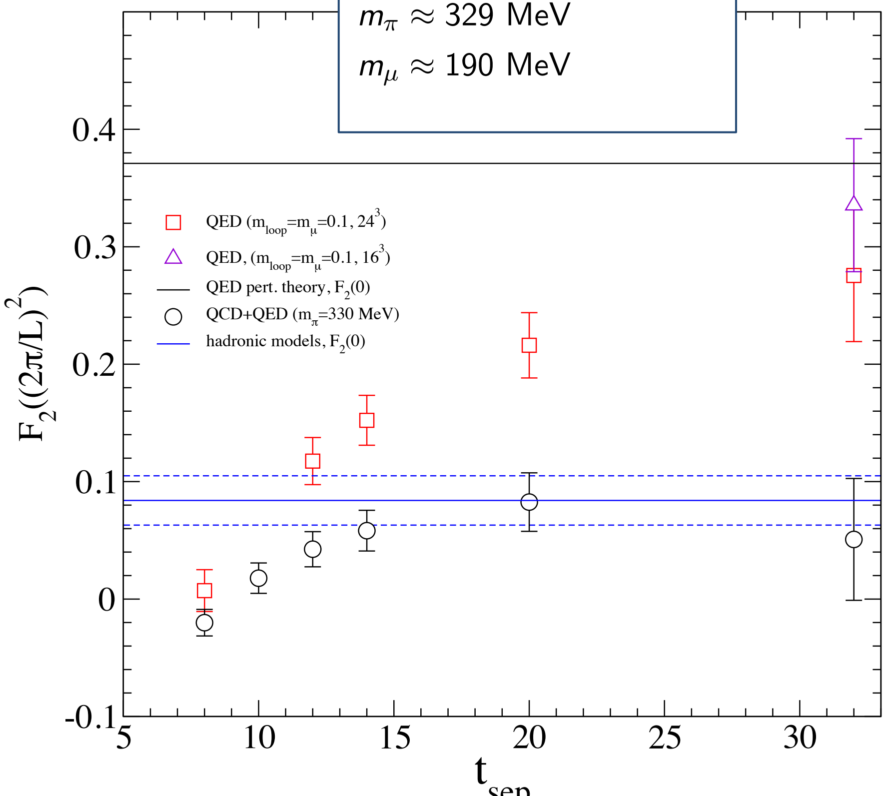


24^3 lattice size

$Q^2 = 0.11$ and $0.18\ \mathrm{GeV}^2$

$m_\pi \approx 329$ MeV

$m_\mu \approx 190$ MeV



- Connected part only

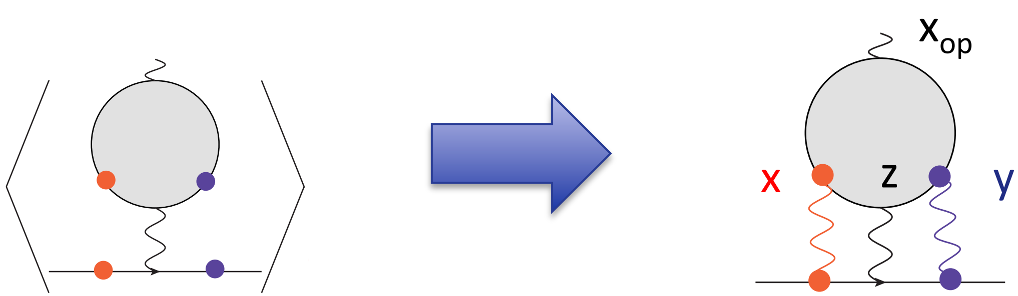
- QED only calculation consistent with QED loop calculation for larger volume

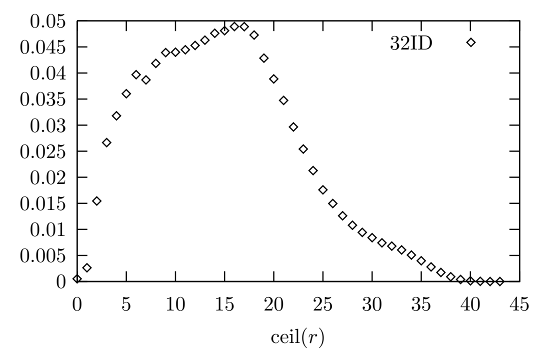
- QED+QCD
  - ball park of model values
  - significant exited state effects ?

Coordinate space Point photon method

[Luchang Jin et al. , arXiv:1510.07100]

- Treat all 3 photon propagators exactly (3 analytical photons) , which makes the quark loop and the lepton line connected :
  disconnected problem in Lattice QED+QCD -> connected problem with analytic photon
- QED 2-loop in coordinate space. Stochastically sample, two of quark-photon vertex location x,y, z and x_{op} is summed over space-time exactly

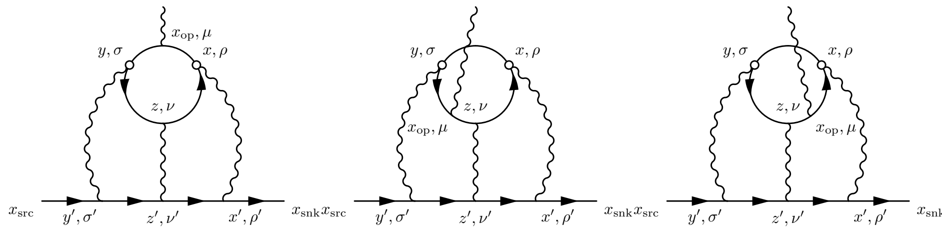




- Short separations, Min[|x-z|,|y-z|,|x-y|] < R ~ O(0.5) fm, which has a large contribution due to confinement, are summed for all pairs
- longer separations, Min[|x-z|,|y-z|,|x-y|] >= R, are done stochastically with a probability shown above (Adaptive Monte Carlo sampling)

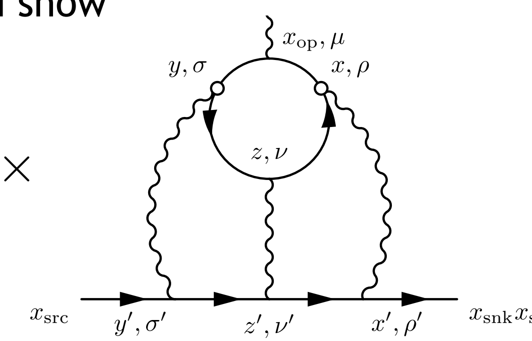
Conserved current & moment method

- [conserved current method at finite q2] To tame UV divergence, one of quark-photon vertex (external current) is set to be conserved current (other three are local currents). All possible insertion are made to realize conservation of external currents config-by-config.



- [moment method , q2→0] By exploiting the translational covariance for fixed external momentum of lepton and external EM field, q->0 limit value is directly computed via the first moment of the relative coordinate, xop – (x+y)/2, one could show

$$\frac{\partial}{\partial q_i}\mathcal{M}_\nu(\vec{q})|_{\vec{q}=0} = i \sum_{x,y,z,x_{\rm op}} (x_{\rm op} - (x+y)/2)_i \times$$

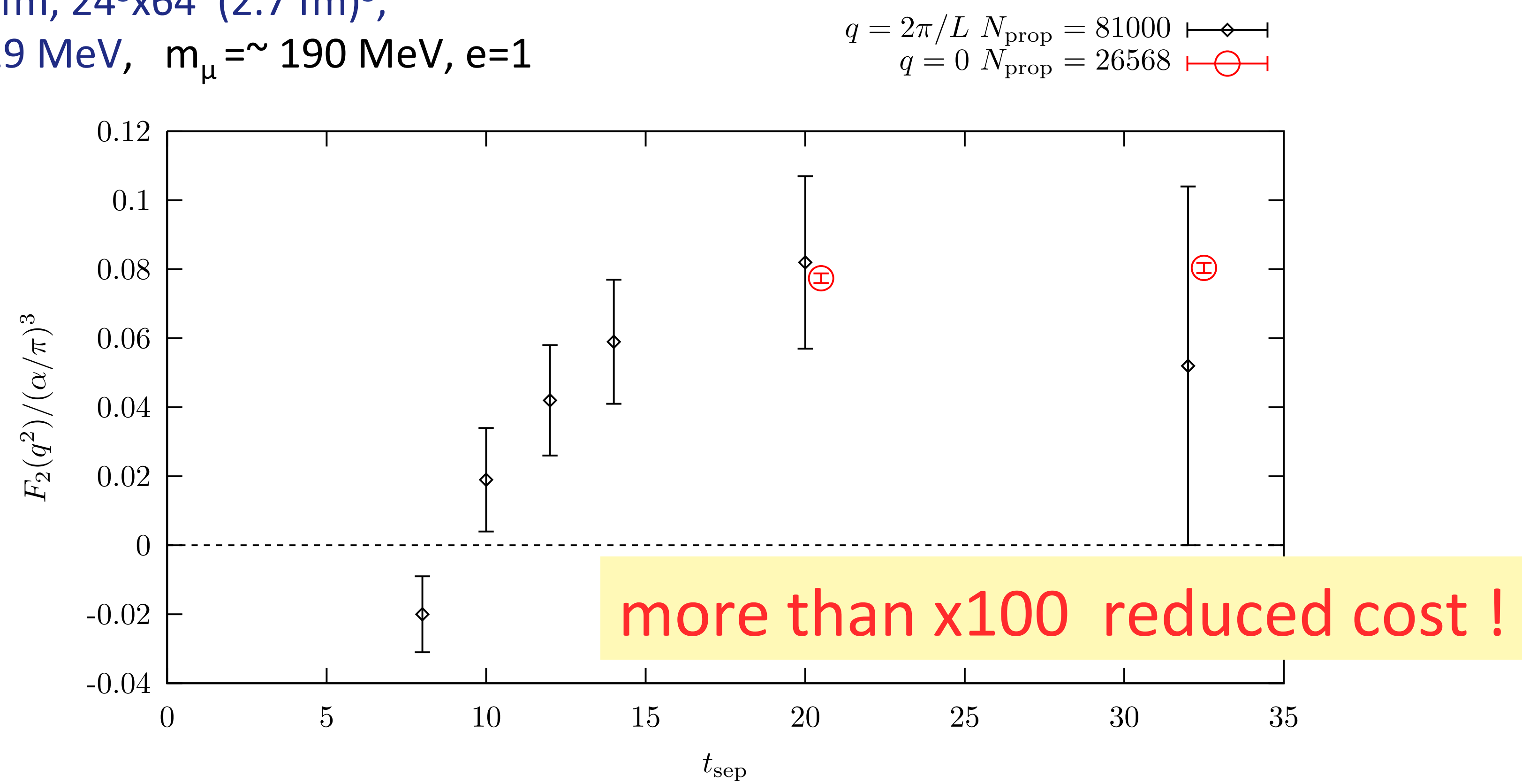


to directly get $F_2(0)$ without extrapolation.

$$\text{Form factor}: \ \Gamma_\mu(q) = \gamma_\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2\, m_l} F_2(q^2)$$

Dramatic Improvement !
Luchang Jin

$a=0.11$ fm, $24^3 \times 64$ $(2.7\text{ fm})^3$,
$m_\pi = 329$ MeV, $m_\mu =\sim 190$ MeV, e=1

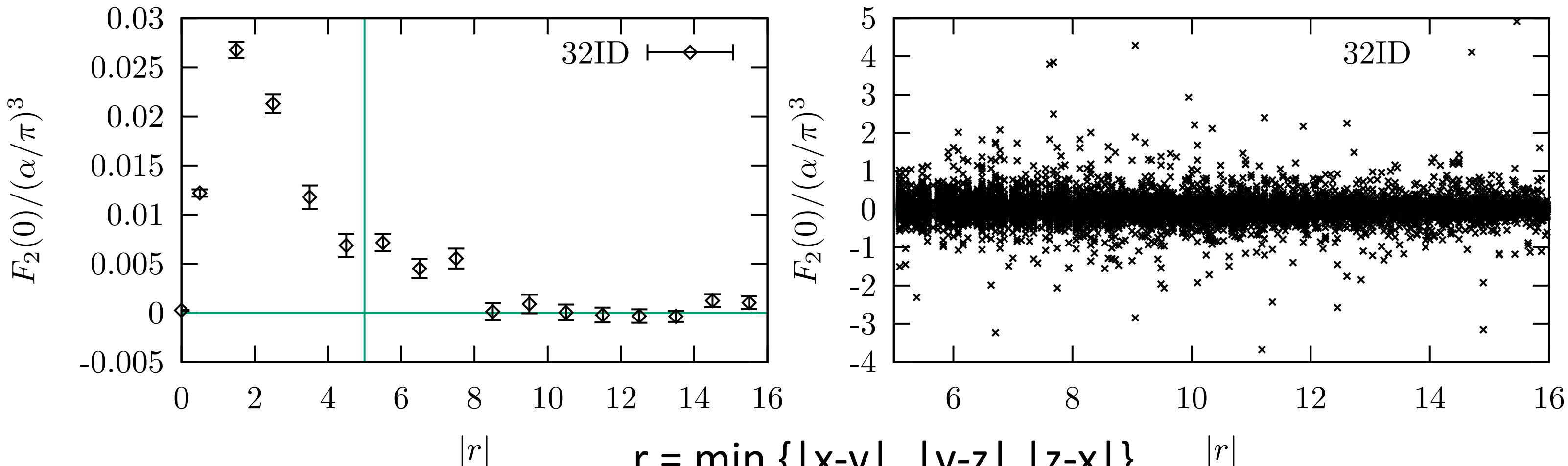


Method	$F_2/(\alpha/\pi)^3$	N_{conf}	N_{prop}	$\sqrt{\mathrm{Var}}$
Conserved	0.0825(32)	12	$(118+128)\times 2\times 7$	0.65
Mom.	0.0804(15)	18	$(118+128)\times 2\times 3$	0.24

M_π=170 MeV cHLbL result

[Luchang Jin et al. , arXiv:1510.07100]

- V=(4.6 fm)^3, a = 0.14 fm, m_μ=130 MeV, 23 conf
- pair-point sampling with AMA (1000 eigV, 100CG) , > 6000 meas/conf
  - |x-y| <= 5, all pairs, x2-5 samples for shorter distances, 217 pairs (10 AMA-exact)
  - |x-y| > 5, 512 pairs (48 AMA-exact)
- 13.2 BG/Q Rack-days



r = min {|x-y|, |y-z|,|z-x|}

$$\frac{(g_\mu - 2)_{\mathrm{cHLbL}}}{2} = (0.1054 \pm 0.0054)(\alpha/\pi)^3 = (132.1 \pm 6.8) \times 10^{-11}.$$

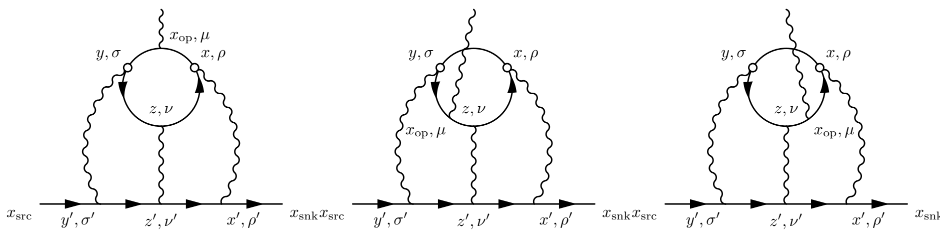
Strange contribution : (0.0011± 0.005) $(\alpha/\pi)^3$

M_{π}=170 MeV cHLbL result (contd.)

"Exact" ... q = 2pi / L,

"Conserved (current)" ... q=2pi/L, 3 diagrams

"Mom" ... moment method q->0, with AMA



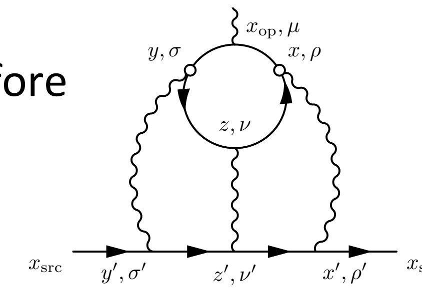
Method	$F_2/(\alpha/\pi)^3$	N_{conf}	N_{prop}	$\sqrt{\mathrm{Var}}$	r_{max}	SD	LD	ind-pair
Exact	0.0693(218)	47	$58 + 8 \times 16$	2.04	3	−0.0152(17)	0.0845(218)	0.0186
Conserved	0.1022(137)	13	$(58 + 8 \times 16) \times 7$	1.78	3	0.0637(34)	0.0385(114)	0.0093
Mom. (approx)	0.0994(29)	23	$(217 + 512) \times 2 \times 4$	1.08	5	0.0791(18)	0.0203(26)	0.0028
Mom. (corr)	0.0060(43)	23	$(10 + 48) \times 2 \times 4$	0.44	2	0.0024(6)	0.0036(44)	0.0045
Mom. (tot)	0.1054(54)	23						

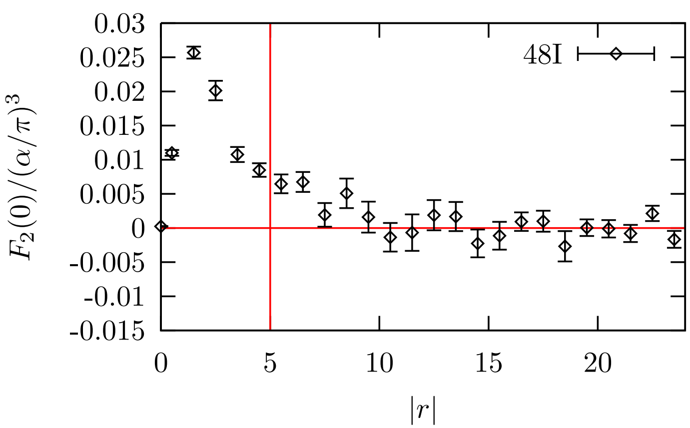
physical M_π=140 MeV cHLbL result

[Luchang Jin et al. , preliminary]

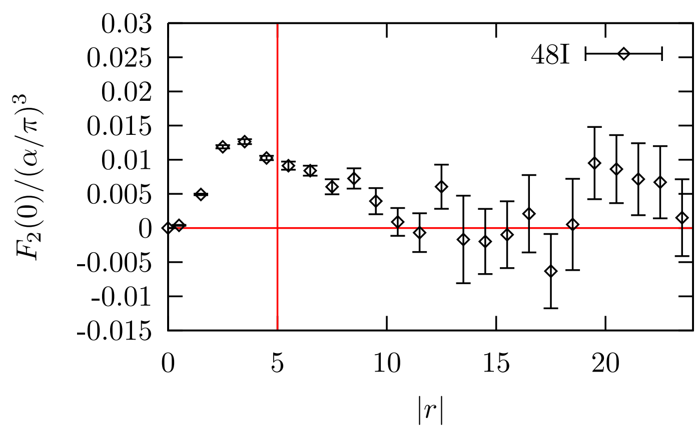
- V=(5.5 fm)^3, a = 0.11 fm, m_μ=106 MeV, 69 conf [RBC/UKQCD]
- Two stage AMA (2000 eigV, 200CG and 400 CG) using zMobius, ~4500 meas/conf
- 160 BG/Q Rack-days

integrand safely suppressed before reaching r ~ L/2





r = min {|x-y|, |y-z|,|z-x|}



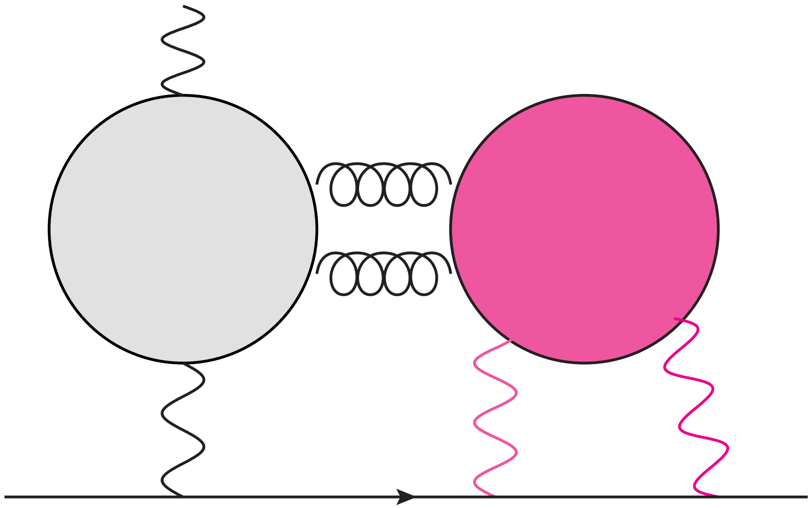
r = max{|x-y|, |y-z|,|z-x|}

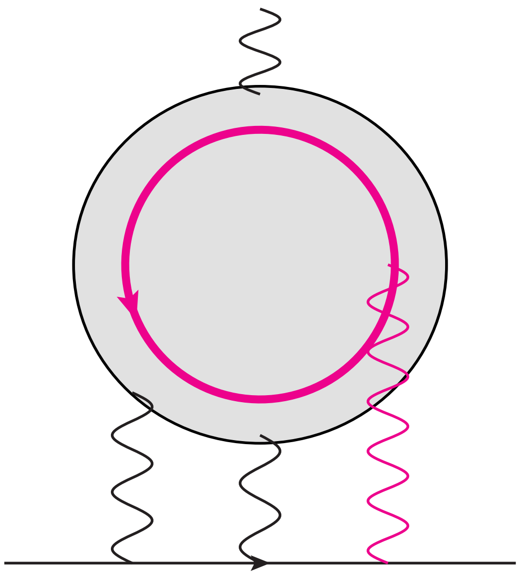
$$\frac{(g_\mu - 2)_{\rm cHLbL}}{2} = (0.933 \pm 0.0073)(\alpha/\pi)^3 = (116.9 \pm 9.1) \times 10^{-11}$$

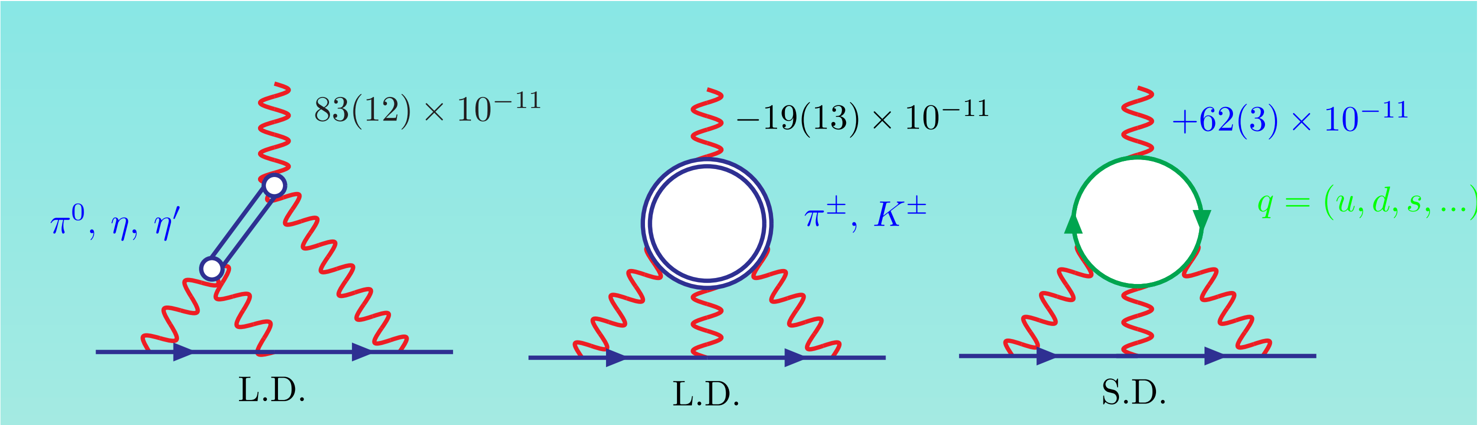
(preliminary, stat err only)

Disconnected diagrams in HLbL

- Disconnected diagrams

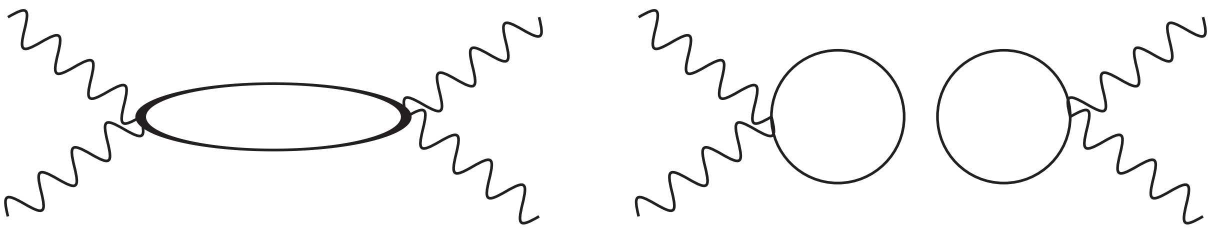






Disconnected HLbL would be non-negligible

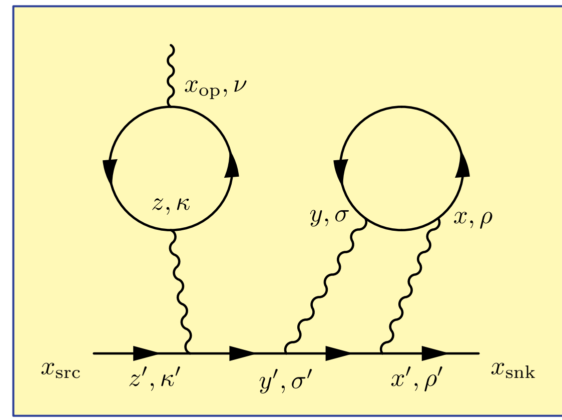
- The major contribution, single pi0 (and η , η ’) exchange diagrams through 2 $\gamma \rightarrow$ π0, would have both connected and disconnected contributions.

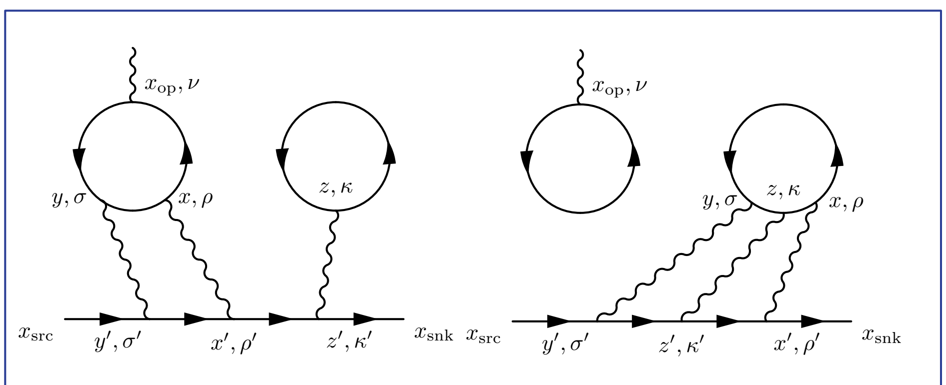


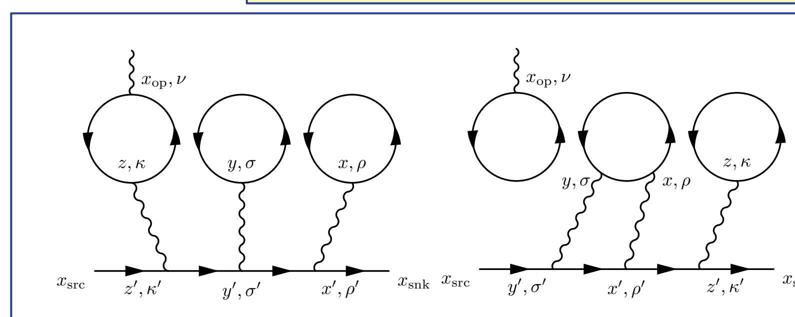
- A quark model consideration for LbL pi0 exchange turns out to be Con : DisCon roughly same size with opposite sign (L. Jin)
- Good news : it’s not η ’ (only), so S/N would not grow exponentially with the propagation length.
- Bad news : it’s disconnected quark loops, and many of them.

SU(3) hierarchies for dHLbL

- At $m_s = m_{ud}$ limit, following type of dHLbL survives due to $Qu + Qd + Qs = 0$
- Physical point run is in progress using similar techniques to cHLbL.
  preliminary result
  a negative value with ~30% stat err !
- $O(m_s - m_{ud})\ /3$ and $O(\ (m_s - m_{ud})^2\ )$







Systematic errors

- Missing disconnected diagrams
  → compute them

- Finite volume

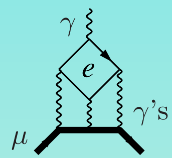
- Discretization error

  → a scaling study for 1/a = 2.7 GeV, 64 cube lattice at physical quark mass is proposed to ALCC at Argonne

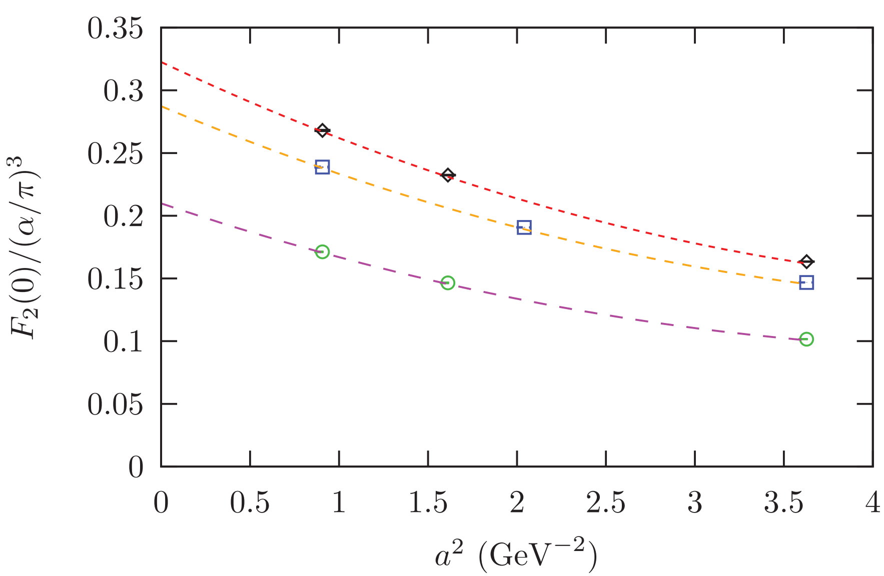
- ...

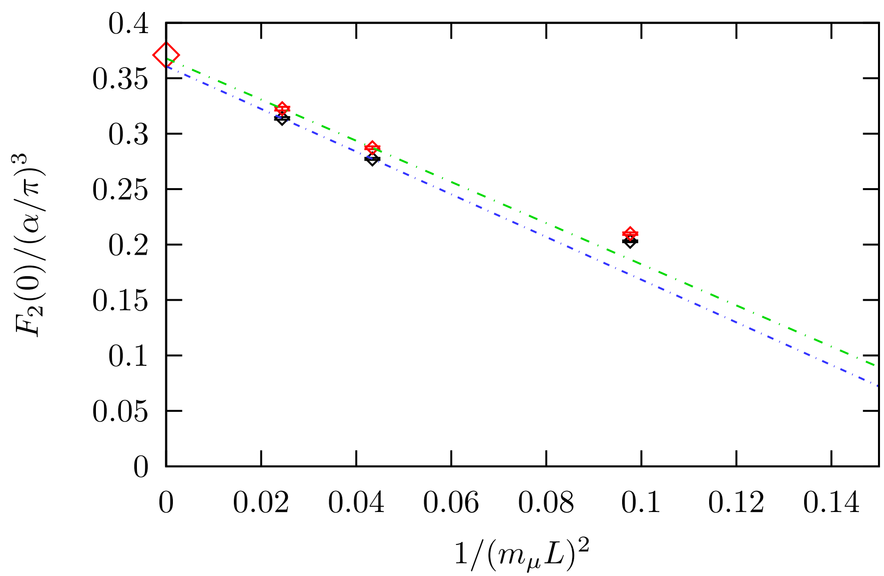
Systematic effects in QED only study

- muon loop, muon line
- a = a m_μ / (106 MeV)
- L= 11.9, 8.9, 5.9 fm
- known result : F2 = 0.371 (diamond) correctly reproduced (good check)



$$a_\mu^{(6)}(\mathrm{lbl}, e) = \left[\frac{2}{3}\pi^2 \ln\frac{m_\mu}{m_e} + \frac{59}{270}\pi^4 - 3\,\zeta(3) - \frac{10}{3}\pi^2 + \frac{2}{3} + O\left(\frac{m_e}{m_\mu}\ln\frac{m_\mu}{m_e}\right)\right]\left(\frac{\alpha}{\pi}\right)^3$$

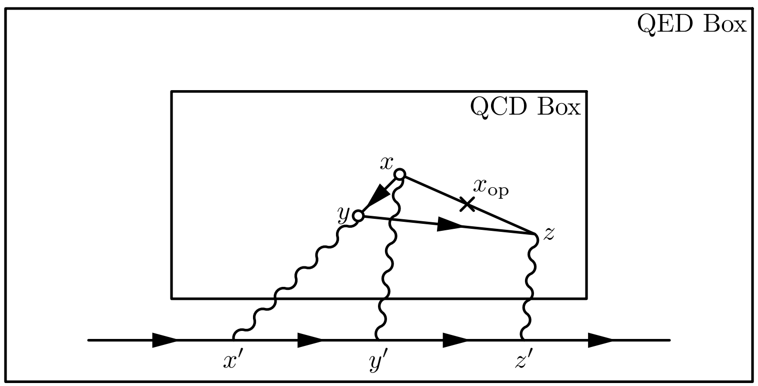


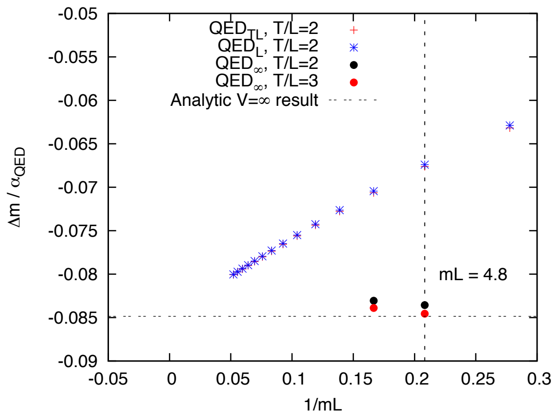


FV and discretization error could be as large as 20-30 %,
similar discretization error seen from QCD+QED study

QCD box in QED box

- FV from quark is exponentially suppressed ~ exp(- M_π L_{QCD})
- Dominant FV effects would be from photon
- Let photon and muon propagate in larger (or infinite) box than that of quark



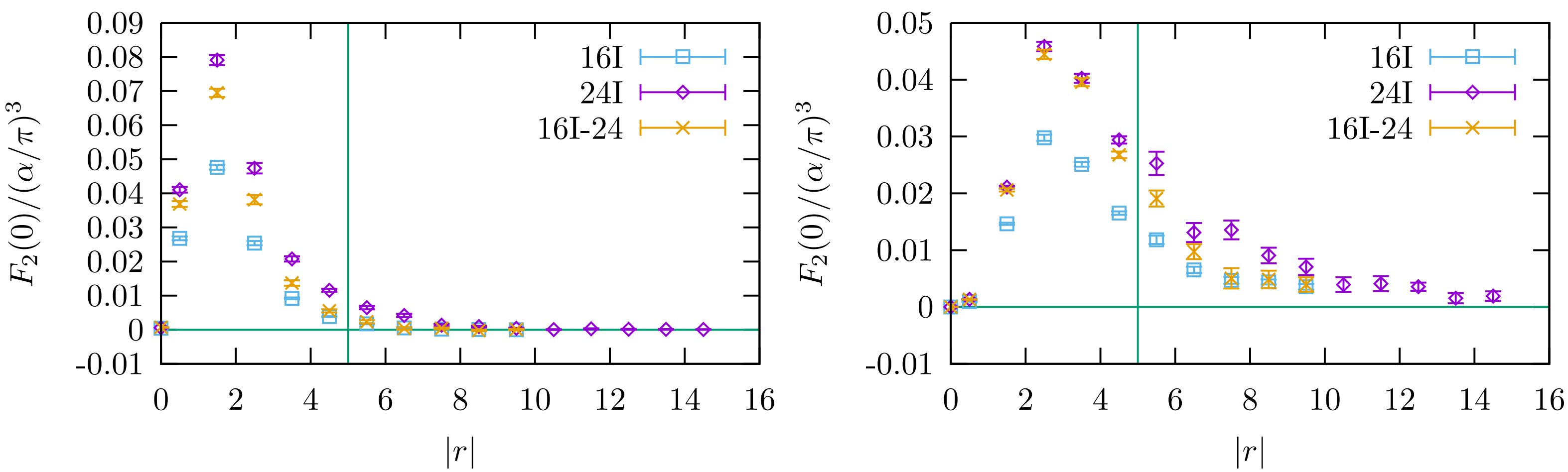


- We could examine different lepton/photon in the off-line manner e.g. QED_L (Hayakwa-Uno 2008) with larger box, Twisting Averaging [Lehner TI LATTICE14] or Infinite Vol. Photon propagators [C. Lehner, L.Jin, TI LATTICE15]

QED box in QCD box (contd.)

- Mπ=420 MeV, mμ=330 MeV, 1/a=1.7 GeV
- $(16)^3$ = $(1.8\ \mathrm{fm})^3$ QCD box in $(24)^3$= $(2.7\ \mathrm{fm})^3$ QED box

Ensemble	$m_\pi L$	QCD Size	QED Size	$\frac{F_2(q^2=0)}{(\alpha/\pi)^3}$
16I	3.87	$16^3 \times 32$	$16^3 \times 32$	$0.1158(8)$
24I	5.81	$24^3 \times 64$	$24^3 \times 64$	$0.2144(27)$
16I-24		$16^3 \times 32$	$24^3 \times 64$	$\boxed{0.1674(22)}$



r = min {|x-y|, |y-z|,|z-x|}   r = max{|x-y|, |y-z|,|z-x|}

Summary

- Connected HLbL calculation is improved very rapidly
- Many orders of magnitudes improvements
  - coordinate-space integral using analytic photon propagator with adaptive probability (point photon method)
  - config-by-config conserved external current
  - take moment of relative coordinate to directly take q→0
  - AMA

→ 8 % stat. error at physical point

(preliminary, stat err only)

$$\frac{(g_\mu - 2)_{\mathrm{cHLbL}}}{2} = (0.933 \pm 0.0073)(\alpha/\pi)^3 = (116.9 \pm 9.1) \times 10^{-11}$$

- SU(3) unsuppressed disconnected diagram has signal also at physical point
- Still large systematic errors (missing disconnected, FV, discretization error, ...)
- Goal : 10% error

Future plans

- (discretization error) Nf=2+1 DWF/ Mobius ensemble at physical point, L=5.5 fm, a=0.083 fm, $(64)^3$ at ALCC @Argonne started to run
- (FV study) QCD box in QED box at physical point
- Disconnected diagrams

Backup slides / for discussion

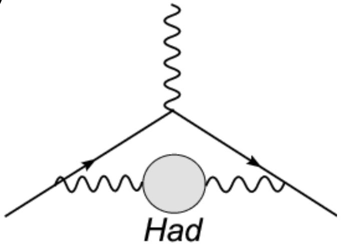
interplays between dispersive approach and Lattice

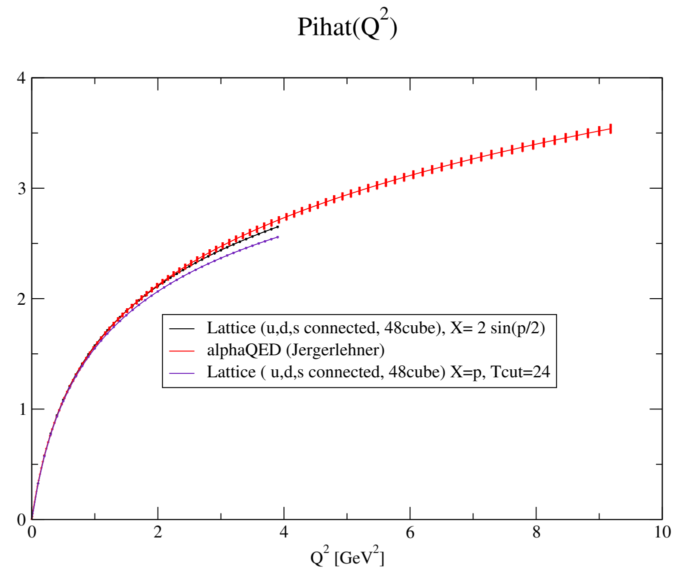
- g-2 HVP
- Vus from strangeness τ inclusive decay

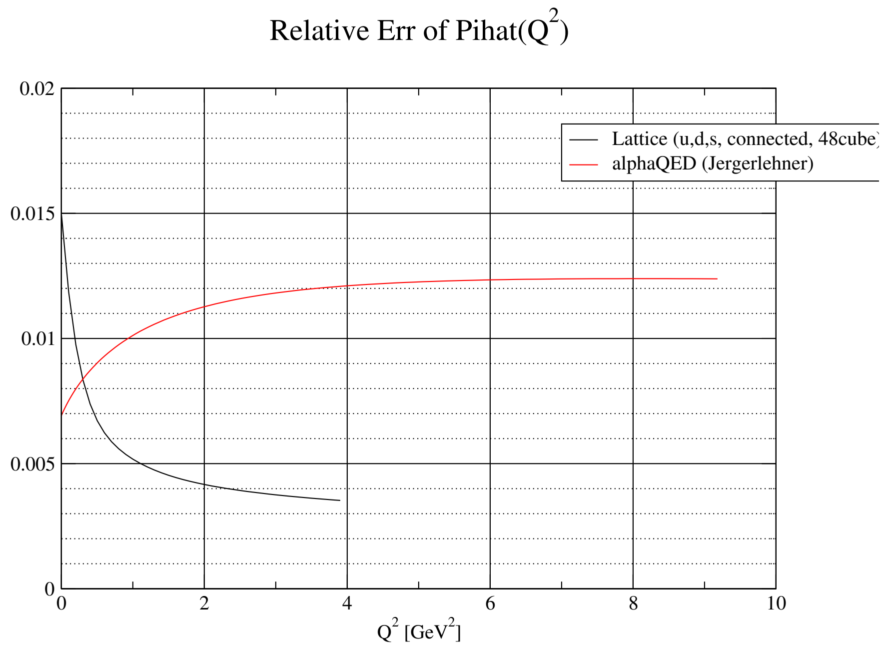
(plan B) Interplays between lattice and dispersive approach g-2

- Dispersive approach from R-ratio R(s)

$$\hat{\Pi}(Q^2) = \frac{Q^2}{3}\int_{s_0} ds \frac{R(s)}{s(s+Q^2)}$$



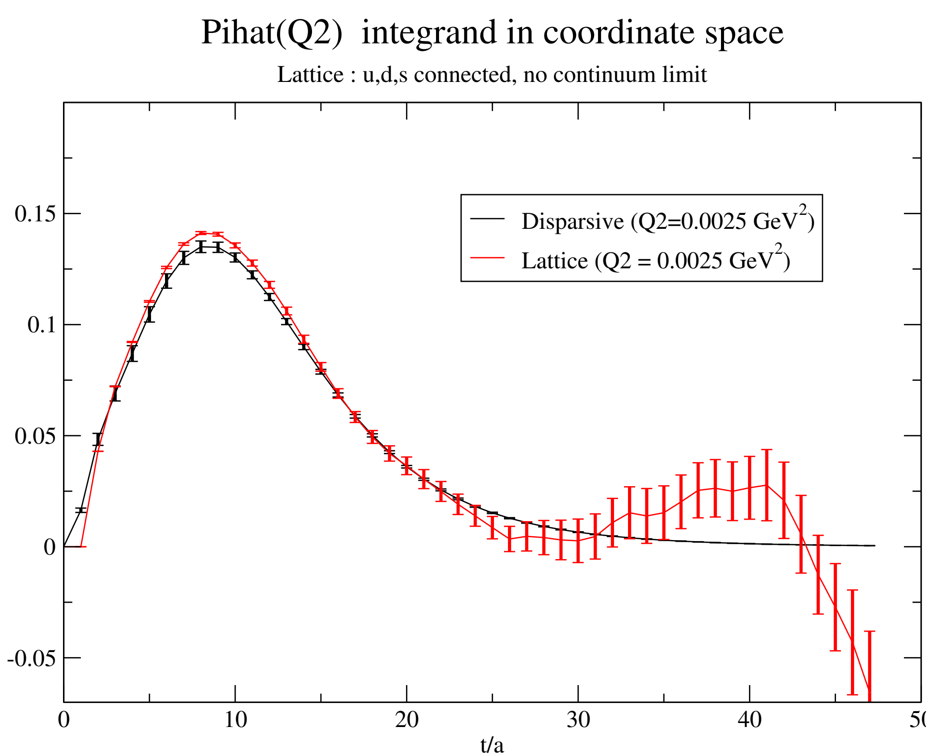


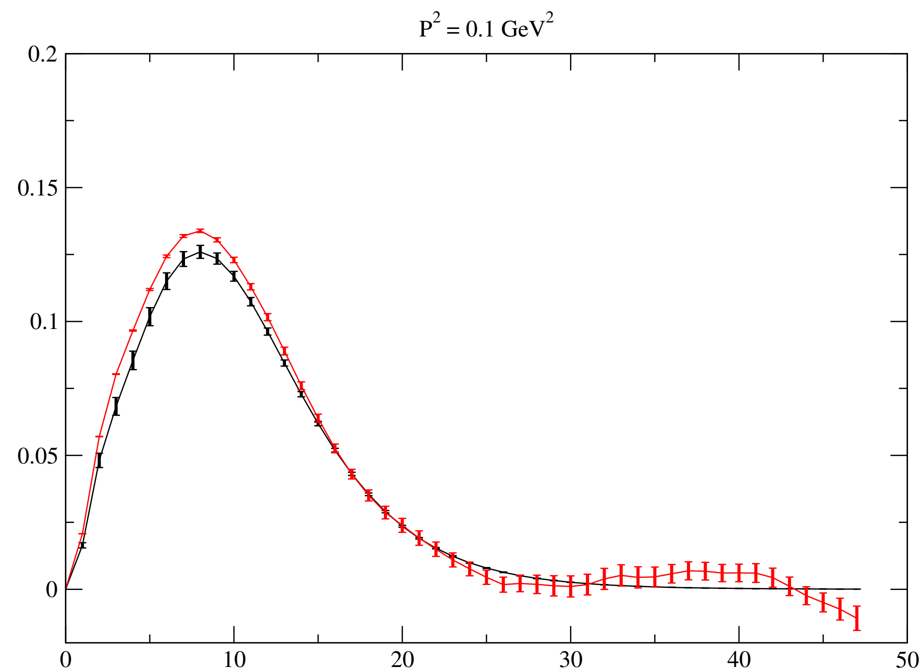


also [ETMC, Mainz, ...]

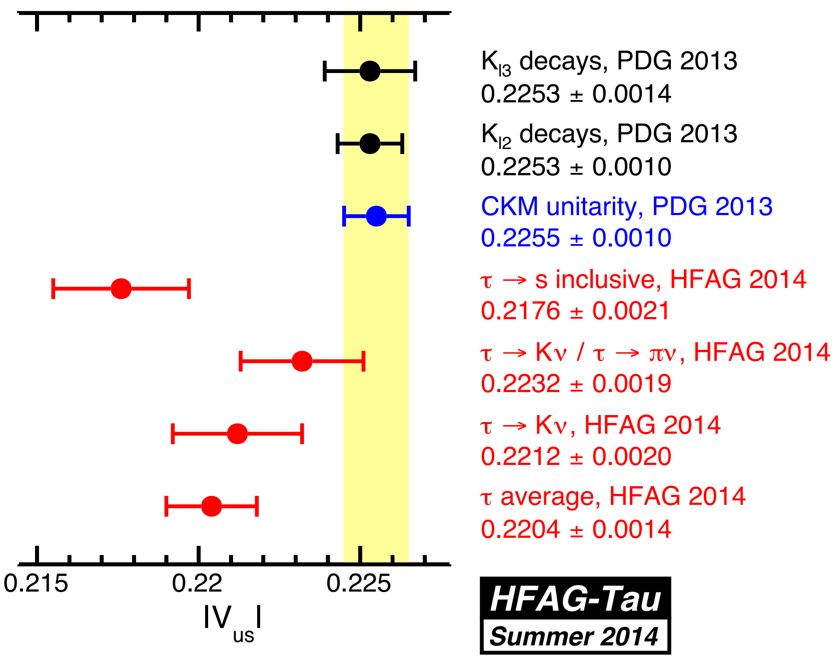
- Can we combine dispersive & lattice and get more precise (g-2)HVP than both ? [2011 Bernecker Meyer]
- Inverse Fourier trans to Euclidean vector correlator
- Relevant for g-2 $Q^2 = (m_\mu/2)^2 = 0.0025\ \mathrm{GeV}^2$
- It may be interesting to think

$$\frac{\hat\Pi(Q^2)}{Q^2} = \left[\frac{\hat\Pi(Q^2)}{Q^2} - \frac{\hat\Pi(P^2)}{P^2}\right]^{\mathrm{Exp}} + \left[\frac{\hat\Pi(P^2)}{P^2}\right]^{\mathrm{Lat}}$$





V_{us} extraction strangeness tau inclusive decay



Tau decay

- $\tau \to \nu + had$ through V-A vertex
- Apply the optical theorem to related to VV and AA hadronic vacuum polarization (HVP)
- For hadrons with strangeness -1, CKM matrix elements V_{us} is multiplied
- ν takes energy away, makes differential cross section is related to the HVPs (c.f. in e^+e^- case, the total cross section is directly related to HVP)

$$
\begin{aligned}
R_{ij} &= \frac{\Gamma(\tau^- \to \text{hadrons}_{ij}\ \nu_\tau)}{\Gamma(\tau^- \to e^-\bar{\nu}_e\nu_\tau)} \\
&= \frac{12\pi|V_{ij}^2|S_{EW}}{m_\tau^2}\int_0^{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)\underbrace{\left[\left(1+2\frac{s}{m_\tau^2}\right)\text{Im}\Pi^{(1)}(s)+\text{Im}\Pi^{(0)}(s)\right]}_{\equiv\ \text{Im}\,\Pi(s)}
\end{aligned}
$$

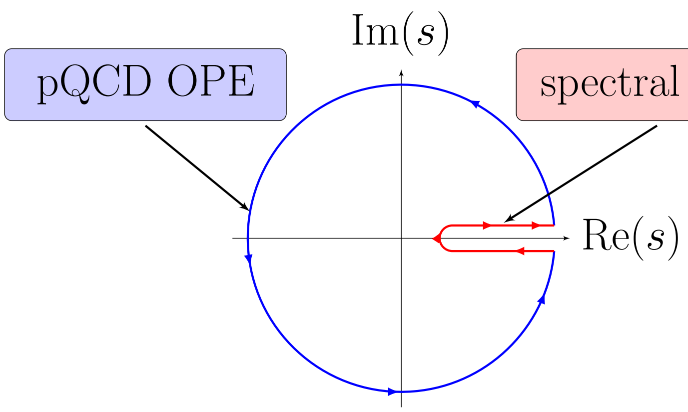
- The Spin=0 and 1, vacuum polarization, Vector(V) or Axial (A) current-current two point

$$
\begin{aligned}
\Pi^{\mu\nu}_{ij;V/A}(q^2) &= i\int d^4x e^{iqx}\left\langle 0|T J^{\mu}_{ij;V/A}(x) J^{\dagger\mu}_{ij;V/A}(0)|0\right\rangle \\
&= (q^\mu q^\nu - q^2 g^{\mu\nu})\Pi^{(1)}_{ij;V/A}(q^2) + q^\mu q^\nu \Pi^{(0)}_{ij;V/A}
\end{aligned}
$$

Finite Energy Sum Rule (FESR)

- Do the finite radius contour integral
- Real axis integral from experimental R_τ
- Use pQCD and OPE for the large circle integral
- Any analytic weight function $w(s)$

$$\int_{s_{th}}^{s_0} \mathrm{Im}\Pi(s)w(s) = \frac{i}{2}\oint_{|s|=s_0} ds\Pi(s)w(s)$$



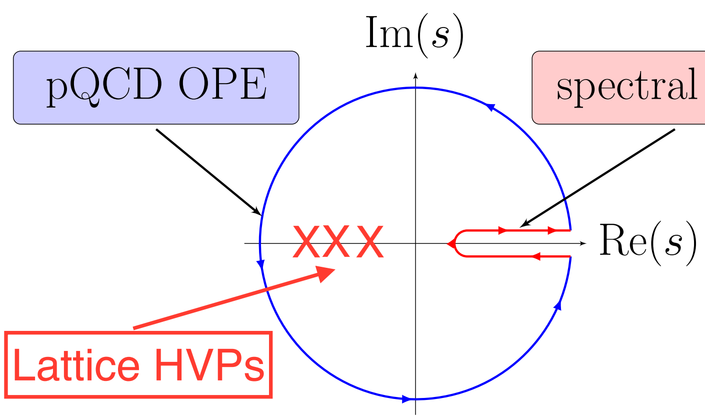
Combining FESR and Lattice

- If we have a reliable estimate for $\Pi(s)$ in Euclidean (space-like) points, $s = -Q_k^2 < 0$, we could extend the FESR with weight function $w(s)$ to have poles there,

$$\int_{s_{th}}^{\infty} w(s)\mathrm{Im}\Pi(s) = \pi \sum_k^{N_p} \mathrm{Res}_k[w(s)\Pi(s)]_{s=-Q_k^2}$$

$$\Pi(s) = \left(1 + 2\frac{s}{m_\tau^2}\right) \mathrm{Im}\Pi^{(1)}(s) + \mathrm{Im}\Pi^{(0)}(s) \propto s \quad (|s| \to \infty)$$

- For $N_p \geq 3$, the $|s| \to \infty$ circle integral vanishes.



weight function $w(s)$

- Example of weight function

$$w(s) = \prod_k^{N_p} \frac{1}{(s+Q_k^2)} = \sum_k a_k \frac{1}{s+Q_k^2}, \quad a_k = \sum_{j \neq k} \frac{1}{Q_k^2 - Q_j^2}$$

$$\Longrightarrow \sum_k (Q_k)^M a_k = 0 \quad (M = 0, 1, \cdots, N_p - 2)$$

- The residue constraints automatically subtracts $\Pi^{(0,1)}(0)$ and $s\Pi^{(1)}(0)$ terms.
- For experimental data, $w(s) \sim 1/s^n, n \geq 3$ suppresses
  - ▷ *larger error from higher multiplicity final states at larger* $s < m_\tau^2$
  - ▷ *uncertanties due to pQCD+OPE at* $m_\tau^2 < s$
- For lattice, Q_k^2 should be not too small to avoid large stat. error, $Q^2 \to 0$ extrapolation, Finite Volume error(?). Also not too larger than m_τ^2 to make the suppression in time-like $0 < s < m_\tau^2$ working.
- Other $w(s)$ could be useful to enhance some region $s > 0$ which may be usable for $(g-2)_\mu$ HVP (?)
- c.f. HPQCD's HVP moments works

Preliminary results

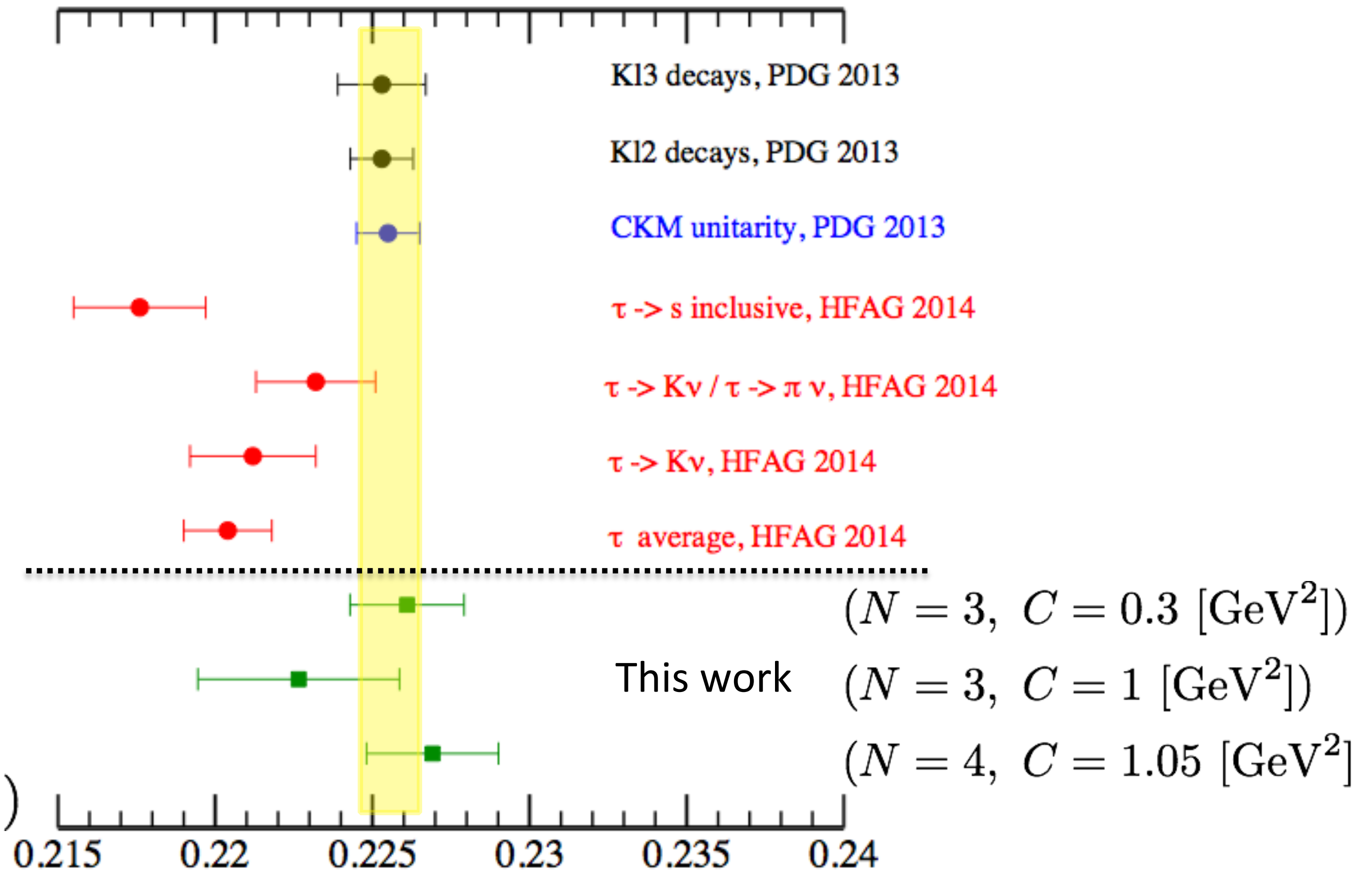
[H. Ohki, A. Juttner, C. Lehner, K. Maltman et al.]

very preliminary

Our result

for all channels

$$(V_1 + V_0 + A_1 + A_0)$$



All our results (C<1, N=3,4) are consistent with each other.

Note : Other systematic errors of sea quark mass chiral extrapolation, lattice O(a^4) discretization,

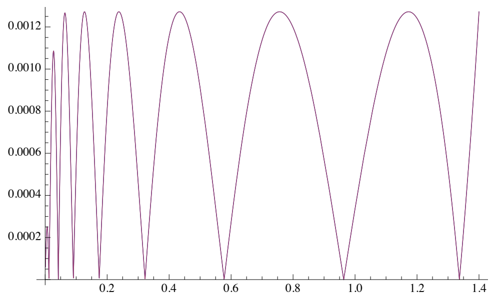
and higher order OPE have not been included. These must be assessed in a future study.

AMA+MADWF(fastPV)+zMobius accelerations

- We utilize complexified 5d hopping term of Mobius action [Brower, Neff, Orginos], zMobius, for a better approximation of the sign function.

$$\epsilon_L(h_M) = \frac{\prod_s^L (1+\omega_s^{-1} h_M) - \prod_s^L (1-\omega_s^{-1} h_M)}{\prod_s^L (1+\omega_s^{-1} h_M) + \prod_s^L (1-\omega_s^{-1} h_M)}, \qquad \omega_s^{-1} = b + c \in \mathbb{C}$$

- 1/a~2 GeV, Ls=48 Shamir ~ Ls=24 Mobius (b=1.5, c=0.5) ~ Ls=10 zMobius (b_s, c_s complex varying) ~5 times saving for cost AND memory



Ls	|eps(48cube) – eps(zMobius)|
6	0.0124
8	0.00127
10	0.000110
12	8.05e-6

- The even/odd preconditioning is optimized (sym2 precondition) to suppress the growth of condition number due to order of magnitudes hierarchy of b_s, c_s [also Neff found this]

$$\texttt{sym2 :} \quad 1 - \kappa_b M_4 M_5^{-1} \kappa_b M_4 M_5^{-1}$$

- Fast Pauli Villars (mf=1) solve, needed for the exact solve of AMA via MADWF (Yin, Mawhinney) is speed up by a factor of 4 or more by Fourier acceleration in 5D [Edward, Heller]
- All in all, sloppy solve compared to the traditional CG is <u>160 times</u> faster on the physical point 48 cube case. And ~<u>100 and 200 times</u> for the 32 cube, Mpi=170 MeV, 140, in this proposal (1,200 eigenV for 32cube) .

$$\underbrace{\frac{20,000}{600}}_{\text{MADWF+zMobius+deflation}} \times \underbrace{\frac{600*32/10}{300}}_{\text{AMA+zMobius}} = 33.3 \times 6.4 = \underline{\text{210 times faster}}$$

[Blum, TI, Shintani PRD 88 (2013) 094503]

Covariant Approximation Averaging (CAA) a new class of Error reduction techniques

Original

$$\mathcal{O} = \mathcal{O}^{(\mathrm{appx})} + \mathcal{O}^{(\mathrm{rest})}$$

Lattice Symmetry

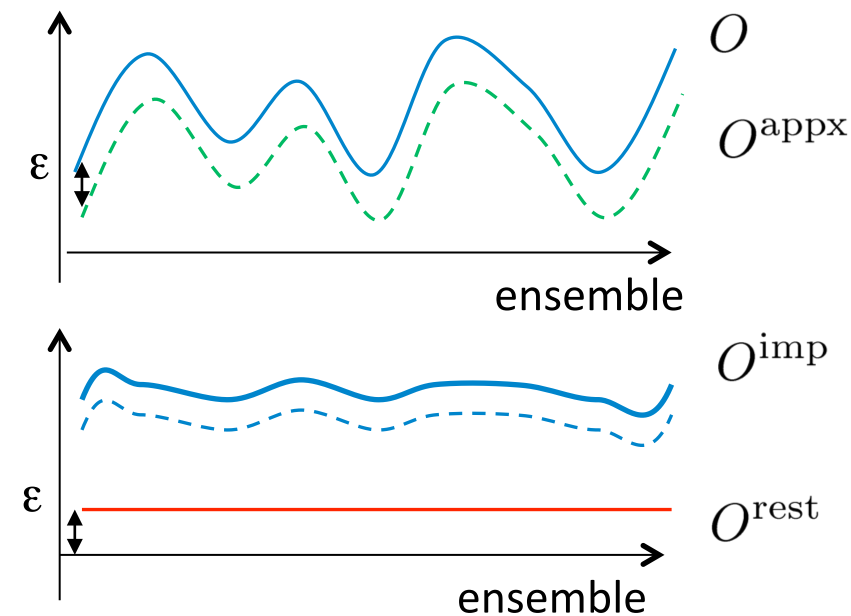
unbiased imporved

$$\mathcal{O}^{(\mathrm{imp})} = \mathcal{O}^{(\mathrm{rest})} + \frac{1}{N_G} \sum_{g \in G} \mathcal{O}^{(\mathrm{appx}),g}$$

Expensive : infrequently measured

Cheap : frequently measured

- $O^{(\mathrm{imp})}$ has smaller error
  $O^{(\mathrm{appx})}$ need to be cheap & not to be too accurate
  N_G suppresses the bulk part of noise cheaply



New bias-free estimator even without covariant approximation by a stochastic choice of source location for the exact/rest computation is now available : Appendix D of arXiv:1402.0244

Examples of Covariant Approximations (contd.)

- All Mode Averaging **AMA**

  Sloppy CG or Polynomial approximations

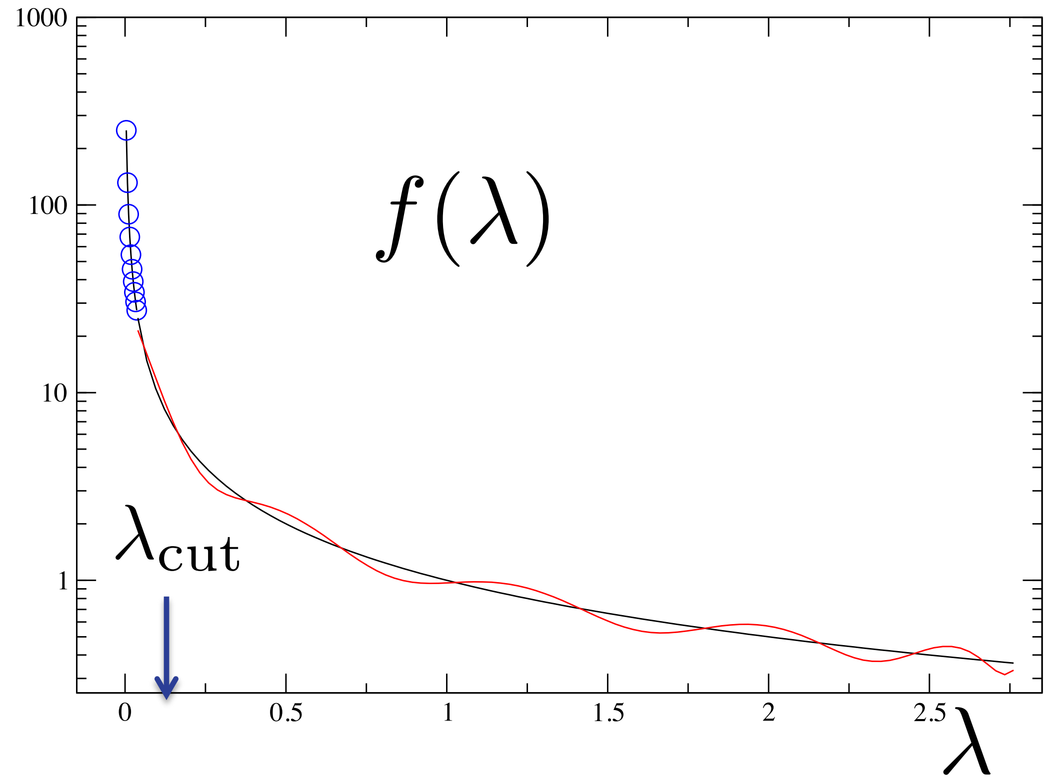
$$\mathcal{O}^{(\mathrm{appx})} = \mathcal{O}[S_l],$$

$$S_l = \sum_\lambda v_\lambda f(\lambda) v_\lambda^\dagger,$$

$$f(\lambda) = \begin{cases} \frac{1}{\lambda}, & |\lambda| < \lambda_{\mathrm{cut}} \\ P_n(\lambda) & |\lambda| > \lambda_{\mathrm{cut}} \end{cases}$$

$$P_n(\lambda) \approx \frac{1}{\lambda}$$

If quark mass is heavy, e.g. ~ strange, low mode isolation may be unneccesary



accuracy control :
- low mode part : # of eig-mode
- mid-high mode : degree of poly.