Hadronic Light-by-Light Scattering and Muon $g-2$: Dispersive Approach

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in collaboration with G. Colangelo, M. Hoferichter and M. Procura

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1. Introduction
2. Lorentz Structure of the HLbL Tensor
3. Master Formula for $(g-2)_\mu$
4. Mandelstam Representation
5. Conclusion and Outlook

1 Introduction

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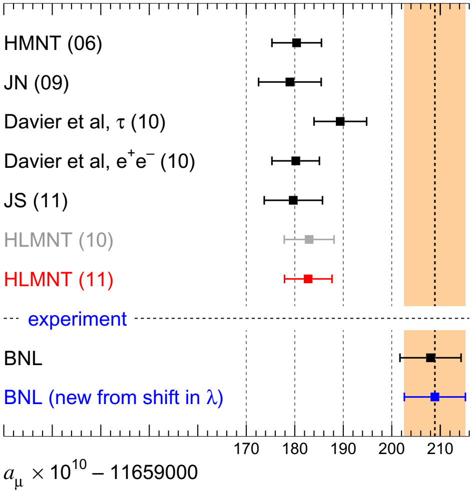
$(g-2)_\mu$: comparison of theory and experiment

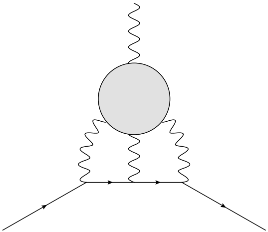


→ Hagiwara et al. 2012

$(g-2)_\mu$: theory vs. experiment

- discrepancy between SM and experiment $\sim 3\sigma$
- hint to new physics?
- new experiments (FNAL, J-PARC) aim at reducing the experimental error by a factor of 4
- theory error completely dominated by hadronic effects
- hadronic vacuum polarisation responsible for largest uncertainty, but will be systematically improved with better data input

Hadronic light-by-light (HLbL) scattering



- up to now only model calculations
- uncertainty estimate based rather on consensus than on a systematic method
- will dominate theory error in a few years

Model calculations of HLbL

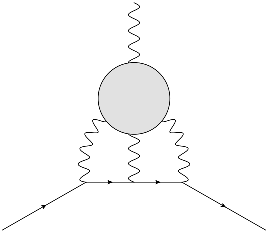
Table 13
Summary of the most recent results for the various contributions to $a_\mu^{\mathrm{LbL;had}} \times 10^{11}$. The last column is our estimate based on our new evaluation for the pseudoscalars and some of the other results.

Contribution	BPP	HKS	KN	MV	BP	PdRV	N/JN
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99±16
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
π, K loops + other subleading in N_c	–	–	–	0±10	–	–	–
axial vectors	2.5±1.0	1.7±1.7	–	22± 5	–	15±10	22± 5
scalars	−6.8±2.0	–	–	–	–	−7± 7	−7± 2
quark loops	21± 3	9.7±11.1	–	–	–	2.3	21± 3
total	83±32	89.6±15.4	80±40	136±25	110±40	105±26	116±39

→ Jegerlehner, Nyffeler 2009

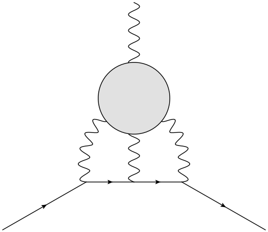
- pseudoscalar pole contribution most important
- pion-loop second most important
- differences between models, large uncertainties

How to improve HLbL calculation?



- lattice QCD making progress
- dispersive approach

Dispersive approach to HLbL



- make use of fundamental principles:
  - gauge invariance, crossing symmetry
  - unitarity, analyticity
- relate HLbL to experimentally accessible quantities

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The HLbL tensor: definitions

- hadronic four-point function:

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3)$$
$$= -i \int dx dy dz e^{-i(q_1 x + q_2 y + q_3 z)} \langle 0 | T j^\mu_{\text{em}}(x) j^\nu_{\text{em}}(y) j^\lambda_{\text{em}}(z) j^\sigma_{\text{em}}(0) | 0 \rangle$$

- EM current:

$$j^\mu_{\text{em}} = \sum_{i=u,d,s} Q_i \bar{q}_i \gamma^\mu q_i$$

- Mandelstam variables:

$s = (q_1 + q_2)^2$, $t = (q_1 + q_3)^2$, $u = (q_2 + q_3)^2$

- for $(g-2)_\mu$, the external photon is on-shell:

$q_4^2 = 0$, where $q_4 = q_1 + q_2 + q_3$

The HLbL tensor

- a priori 138 'naive' Lorentz structures:

$$\begin{aligned}\Pi^{\mu\nu\lambda\sigma} &= g^{\mu\nu}g^{\lambda\sigma}\Pi^1 + g^{\mu\lambda}g^{\nu\sigma}\Pi^2 + g^{\mu\sigma}g^{\nu\lambda}\Pi^3 \\ &\quad + \sum_{i,k,l,m} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma \Pi^4_{ijkl} \\ &\quad + \sum_{i,j} g^{\lambda\sigma} q_i^\mu q_j^\nu \Pi^5_{ij} + \dots\end{aligned}$$

- in 4 space-time dimensions: 2 linear relations among the 138 Lorentz structures → Eichmann et al., 2014
- six dynamical variables, e.g. two Mandelstam variables s, t and the photon virtualities q_1^2, q_2^2, q_3^2, q_4^2

HLbL tensor: gauge invariance

- Ward identities

$$\{q_1^\mu, q_2^\nu, q_3^\lambda, q_4^\sigma\}\Pi_{\mu\nu\lambda\sigma} = 0$$

  imply 95 linear relations between scalar functions Π_i
- off-shell basis: 138 − 95 − 2 = 41 structures
- corresponding to 41 helicity amplitudes
- relations between Π_i imply kinematic zeros

HLbL tensor: Lorentz decomposition

Problem: find a decomposition

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_i T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u; q_j^2)$$

with the following properties:

- Lorentz structures $T_i^{\mu\nu\lambda\sigma}$ manifestly gauge invariant:

$$\{q_1^\mu, q_2^\nu, q_3^\lambda, q_4^\sigma\} T^i_{\mu\nu\lambda\sigma} = 0$$

- scalar functions Π_i free of kinematic singularities and zeros

HLbL tensor: Lorentz decomposition

Recipe by Bardeen, Tung (1968) and Tarrach (1975):

- construct gauge projectors:

$$I_{12}^{\mu\nu} = g^{\mu\nu} - \frac{q_2^\mu q_1^\nu}{q_1 \cdot q_2}, \quad I_{34}^{\lambda\sigma} = g^{\lambda\sigma} - \frac{q_4^\lambda q_3^\sigma}{q_3 \cdot q_4}$$

- gauge invariant themselves, e.g.

$$q_1^\mu I^{12}_{\mu\nu} = 0$$

- leave HLbL tensor invariant, e.g.

$$I_{12}^{\mu\mu'} \Pi_{\mu'\nu\lambda\sigma} = \Pi^{\mu}{}_{\nu\lambda\sigma}$$

HLbL tensor: Lorentz decomposition

Following Bardeen, Tung (1968):

- apply gauge projectors to the 138 initial structures: 95 immediately project to 0
- remove $1/q_1 \cdot q_2$ and $1/q_3 \cdot q_4$ poles by taking appropriate linear combinations
- BT basis: degenerate in the limits $q_1 \cdot q_2 \to 0$, $q_3 \cdot q_4 \to 0$

HLbL tensor: Lorentz decomposition

According to Tarrach (1975):

- degeneracies in the limits $q_1 \cdot q_2 \to 0$, $q_3 \cdot q_4 \to 0$:

$$\sum_k c_k^i T_k^{\mu\nu\lambda\sigma} = q_1 \cdot q_2 X_i^{\mu\nu\lambda\sigma} + q_3 \cdot q_4 Y_i^{\mu\nu\lambda\sigma}$$

- extend basis by additional structures $X_i^{\mu\nu\lambda\sigma}$, $Y_i^{\mu\nu\lambda\sigma}$ taking care of remaining kinematic singularities
- equivalent: implementing crossing symmetry

HLbL tensor: Lorentz decomposition

Solution for the Lorentz decomposition:

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u; q_j^2)$$

- Lorentz structures manifestly gauge invariant
- crossing symmetry manifest: only 7 distinct structures, 47 follow from crossing
- scalar functions Π_i free of kinematic singularities $\Rightarrow$ ideal quantities for a dispersive treatment

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Master formula: contribution to $(g-2)_\mu$

- from gauge invariance:

$$\Pi_{\mu\nu\lambda\rho} = -q_4^\sigma \frac{\partial}{\partial q_4^\rho} \Pi_{\mu\nu\lambda\sigma}$$

- for $(g-2)_\mu$: afterwards take $q_4 \to 0$
- no kinematic singularities in scalar functions: perform these steps with the derived Lorentz decomposition
- only 12 linear combinations of the scalar functions Π_i contribute to $(g-2)_\mu$

Master formula: contribution to $(g-2)_\mu$

$$a_\mu^{\mathrm{HLbL}} = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p) \hat{\Pi}_i(q_1, q_2, -q_1 - q_2)}{q_1^2 q_2^2 (q_1 + q_2)^2 [(p+q_1)^2 - m_\mu^2][(p-q_2)^2 - m_\mu^2]}$$

- $\hat{T}_i$: known integration kernel functions
- five loop integrals can be performed with Gegenbauer polynomial techniques
- Wick rotation possible even in the presence of anomalous thresholds

Master formula: contribution to $(g-2)_\mu$

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2} Q_1^3 Q_2^3 \times \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau),$$

- T_i: known integration kernels
- $\bar{\Pi}_i$: linear combinations of the scalar functions Π_i
- Euclidean momenta: $Q_i^2 = -q_i^2$
- $Q_3^2 = Q_1^2 + Q_2^2 + 2Q_1Q_2\tau$

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Analytic properties of scalar functions

- right- and left-hand cuts in each Mandelstam variable
- double-spectral regions (box topologies)
- anomalous thresholds for large photon virtualities

Mandelstam representation

- we limit ourselves to intermediate states of at most two pions
- writing down a double-spectral (Mandelstam) representation allows us to split up the HLbL tensor:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \dots$$

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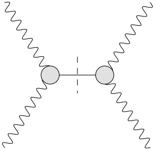
one-pion intermediate state:



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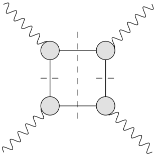
two-pion intermediate state in both channels:



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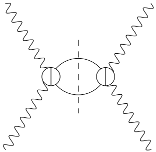
two-pion intermediate state in first channel: 

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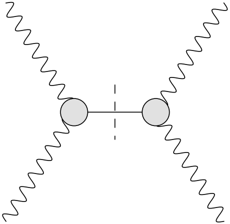
neglected so far: higher intermediate states

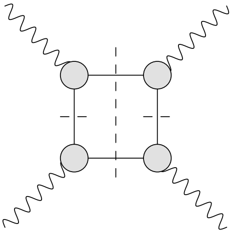
Pion pole



- input: doubly-virtual and singly-virtual pion transition form factors $\mathcal{F}_{\gamma^*\gamma^*\pi^0}$ and $\mathcal{F}_{\gamma^*\gamma\pi^0}$
- dispersive analysis of transition form factor:

$\to$ Hoferichter et al., EPJC **74** (2014) 3180

Pion box



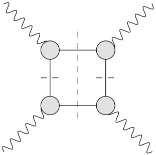
- simultaneous two-pion cuts in two channels
- Mandelstam representation explicitly constructed

$$\Pi_i = \frac{1}{\pi^2}\int ds'dt'\,\frac{\rho_i^{st}(s',t')}{(s'-s)(t'-t)} + (t \leftrightarrow u) + (s \leftrightarrow u)$$

- q^2-dependence: pion vector form factors $F_\pi^V(q_i^2)$ for each off-shell photon factor out

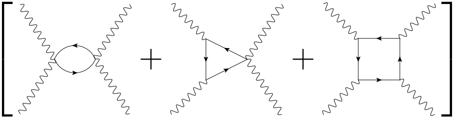
Pion box

- sQED loop projected on BTT basis fulfils the same Mandelstam representation
- only difference are factors of F_π^V
- $\Rightarrow$ box topologies are identical to FsQED:



$$\equiv F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2)$$

$$\times \Big[\quad \Big]$$



- model-independent definition of pion loop

Pion box

Very simple expressions for box contributions in terms of Feynman parameter integrals

$$\Pi_i^{\pi\text{-box}}(q_1^2, q_2^2, q_3^2) = F_\pi^V(q_1^2) F_\pi^V(q_2^2) F_\pi^V(q_3^2) \\ \times \frac{1}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy\, I_i(x, y),$$

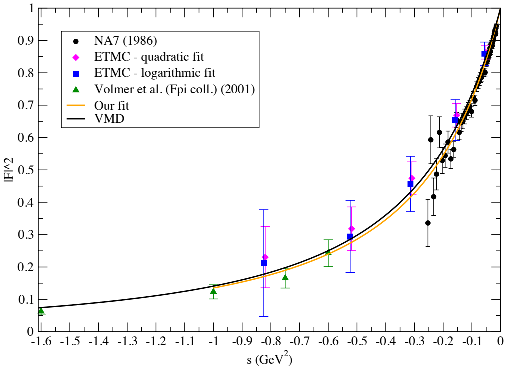
with e.g.

$$I_7(x, y) = -\frac{4}{3} \frac{(1-2x)^2(1-2y)^2 y(1-y)}{\Delta_{123}^3},$$

$$\Delta_{ijk} = M_\pi^2 - xyq_i^2 - x(1-x-y)q_j^2 - y(1-x-y)q_k^2.$$

Pion box

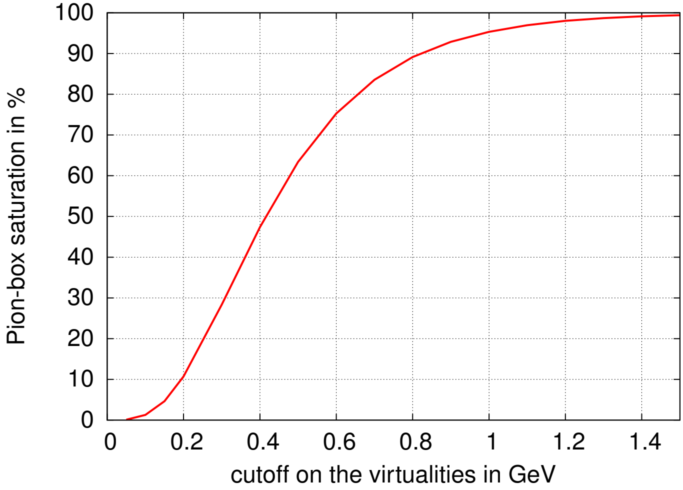
Pion vector form factor in the space-like region:



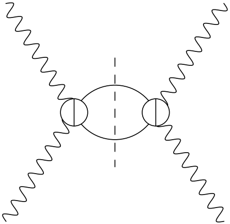
Preliminary results:

$$a_\mu^{\pi\text{-box}} = -15.9 \cdot 10^{-11}, \qquad a_\mu^{\pi\text{-box, VMD}} = -16.4 \cdot 10^{-11}$$

Pion-box saturation with photon virtualities

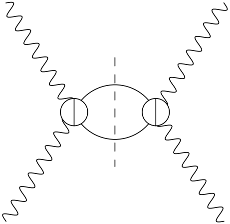


Rescattering contribution



- neglect left-hand cut due to multi-particle intermediate states in crossed channel
- two-pion cut in only one channel
- expansion into partial waves

Rescattering contribution



- unitarity relates it to the helicity amplitudes of the subprocess $\gamma^*\gamma^{(*)} \to \pi\pi$
- dispersive integrals over the imaginary parts allow the reconstruction of $\bar{\Pi}_{\mu\nu\lambda\sigma}$
- sum rules ensure cancellation of unphysical helicity amplitudes

The subprocess

Helicity amplitudes for $\gamma^*\gamma^* \to \pi\pi$: dispersive solution as Roy-Steiner equations

- $\gamma\gamma \to \pi\pi$: $\to$ Moussallam 2010, Hoferichter, Phillips, Schat 2011
- $\gamma^*\gamma \to \pi\pi$: $\to$ Moussallam 2013
- $\gamma^*\gamma^* \to \pi\pi$: work in progress

  $\to$ Hoferichter, Colangelo, Procura, PS 2013

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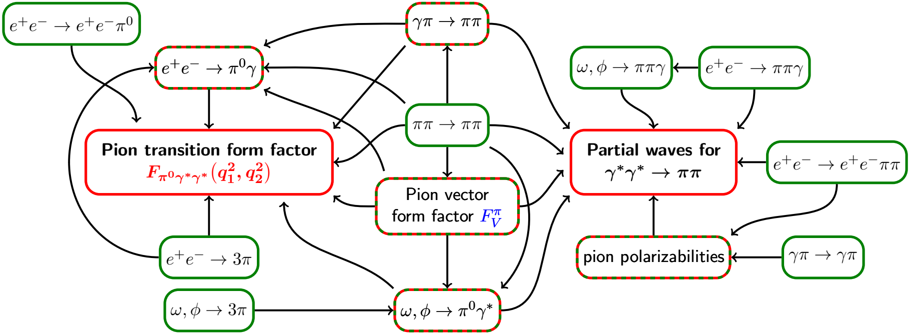
Summary

- our dispersive approach to HLbL scattering is based on fundamental principles:
  - gauge invariance, crossing symmetry
  - unitarity, analyticity
- we take into account the lowest intermediate states: π^0-pole and $\pi\pi$-cuts
- relation to experimentally accessible (or again with data dispersively reconstructed) quantities
- a step towards a model-independent calculation of a_μ

A roadmap for HLbL

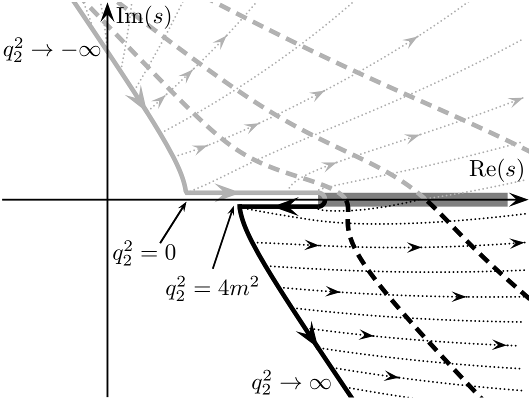


$\to$ Flowchart by M. Hoferichter

Backup

Wick rotation

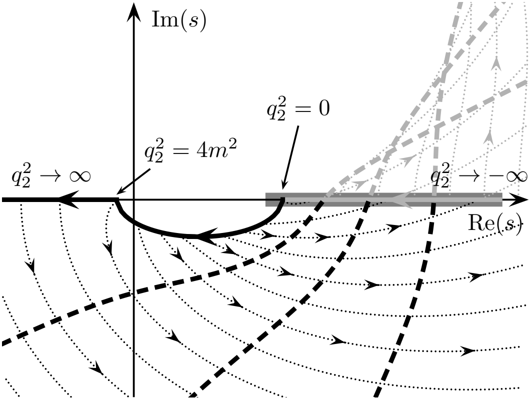
Trajectory of triangle anomalous threshold:



$$0 < q_1^2 < 4m^2$$

Wick rotation

Trajectory of triangle anomalous threshold:



$$4m^2 < q_1^2$$