Renormalons in heavy quark physics and lattice: the pole mass and the gluon condensate

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Symposium on EFTs and LGT,
Munich TUM IAS, May 18-21, 2016

Renormalons

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Renormalon: summation of "bubbles". Running of α.

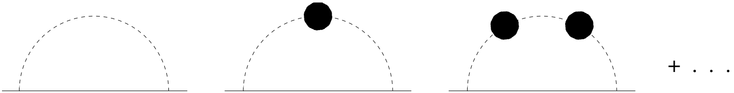

Figure: Sum of the bubbles in the quark propagator.

Pole mass (Bigi, Shifman, Uraltsev, Vainshtein; Beneke, Braun)

$$m_{\rm OS} = m_{\overline{\rm MS}}(1 + B_1\alpha_s + B_2\alpha_s^2 + \cdots) \qquad B_n \sim n!$$

$$\delta m \propto \int^{\mu} dk\, \alpha(k) \sim \alpha_s(\mu)\sum_{n=0}^{\infty}\left(\frac{\beta_0\alpha(\mu)}{2\pi}\right)^n \int^{\mu} dk\, \ln^n\frac{\mu}{k} = \alpha_s(\mu)\sum_{n=0}^{\infty}\left(\frac{\beta_0\alpha(\mu)}{2\pi}\right)^n n!$$

$\beta_0^{\rm QED} = -\frac{4}{3}T_F n_l \to \beta_0^{\rm QCD} = \frac{11}{3}C_A - \frac{4}{3}T_F n_l$ naive non-abelianization.

Beyond bubbles: renormalization group methods (Parisi; Beneke, ...)
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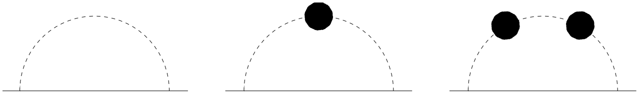

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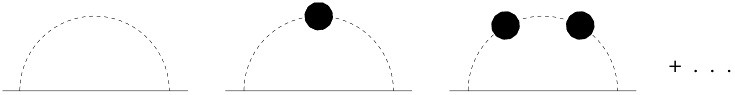

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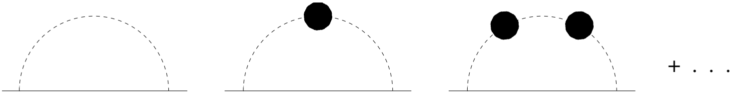

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Modern view: EFT/Factorization

$$\mathcal{L} = \sum_n \frac{1}{m^n} c_n O_n \qquad c(\nu) = \bar{c} + \sum_{n=0}^{\infty} c_n \alpha_s^{n+1}.$$

The Wilson coefficients are believed to be asymptotic: $c_n \sim n!$

IF SO such behavior should comply with the Operator Product Expansion.

EFT/factorization definition of renormalon: Asymptotic behavior of the perturbative expansion such that the associated ambiguity in the summation of the perturbative series can be absorbed into a higher order operator.

Example:

$$M_B = m_{\rm OS} + \bar{\Lambda}_B + \mathcal{O}(1/m_{\rm OS})$$

M_B is renormalon free. Therefore $m_{\rm OS}$ suffers from renormalon ambiguities:

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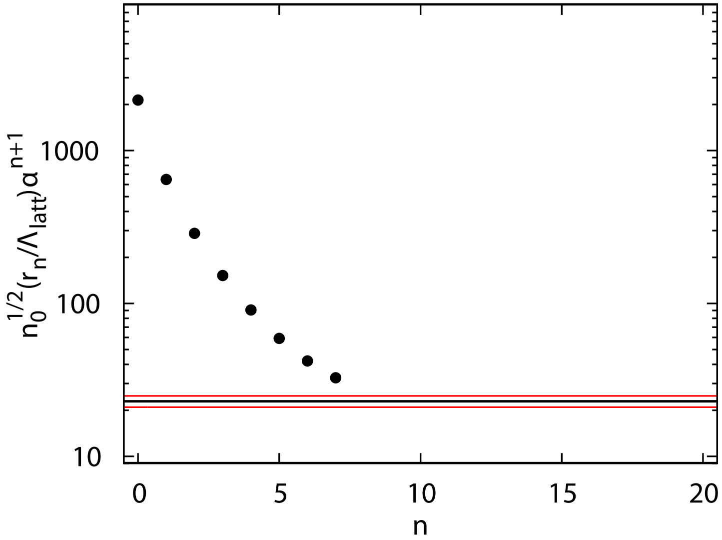

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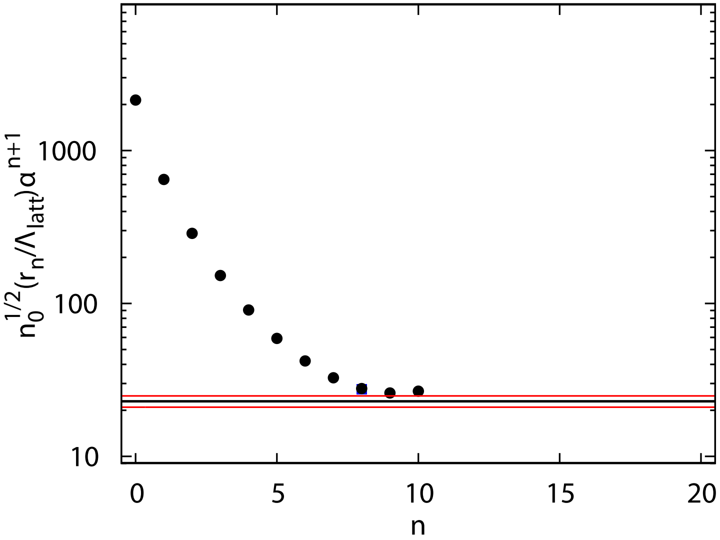
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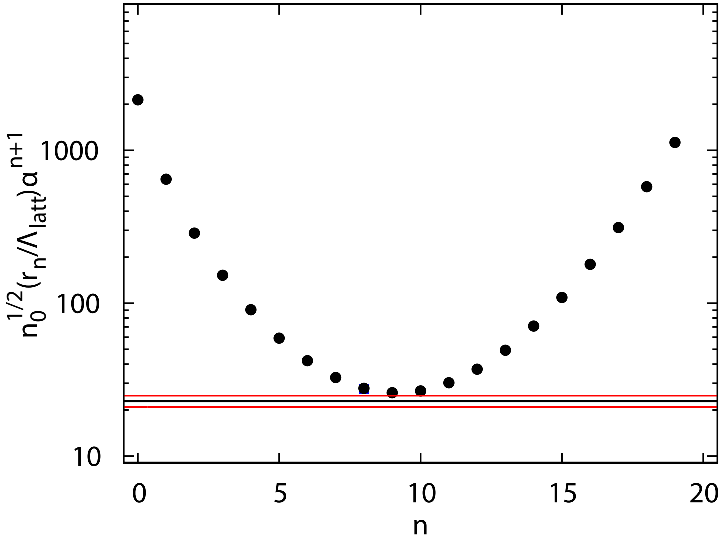

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Maximal accuracy of the Wilson coefficients from a perturbative calculation is (roughly) of the order of

$$\delta c \sim r_{n^*} \alpha_s^{n^*} \,,$$

where $n^* \sim \frac{a}{\alpha_s}$. If a is positive c suffers from a non-perturbative ambiguity of order

$$\delta c \sim (\Lambda_{\rm QCD})^{\frac{|a|\beta_0}{2\pi}} \,.$$

The Borel transform of $c(\nu)$ reads

$$B[c](t) \equiv \sum_{n=0}^{\infty} c_n \frac{t^n}{n!},$$

and c is written in terms of its Borel transform as

$$c = \bar{c} + \int_0^\infty dt\, e^{-t/\alpha_s} B[c](t).$$

The ambiguities in the Wilson coefficient ($c_n \sim n!$) reflects in poles in the Borel transform. If we take the one closest to the origin,

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Current-current correlator ($c_{n*}\alpha_s^{n*+1} \sim \Lambda_{\rm QCD}^4/Q^4 \to c_n \sim n!$):

$$\int d^4x e^{iqx} \langle vac|J(x)J(0)|vac\rangle = (Pert.\ th.) + \frac{\Lambda_{\rm QCD}^4}{Q^4} + \cdots$$

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$$\langle P\rangle = (Pert.\ th.) + a^4\Lambda_{\rm QCD}^4 + \cdots$$

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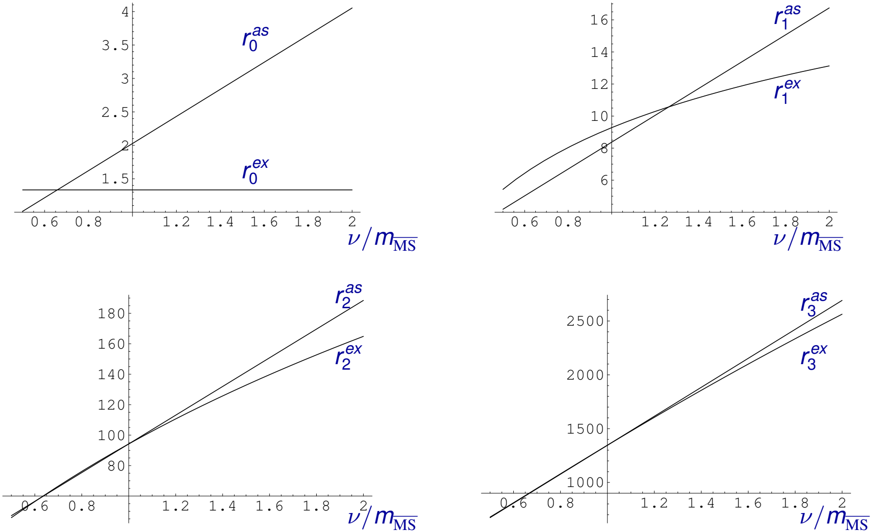
$$\text{Check :} \qquad r_n \overset{n\to\infty}{\sim} m_{\overline{\rm MS}} \left(\frac{\beta_0}{2\pi}\right)^n n! N_m \sum_{s=0}^{n} \frac{\ln^s[\nu/m_{\overline{\rm MS}}]}{s!} \sim \nu$$



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POLYAKOV LOOP versus δm (and m)

Possible to compute the energy of an static source in the lattice: δm of HQET.
We use Numerical Stochastic Perturbation Theory (Di Renzo et al.).

$$L^{(R)}(N_S, N_T) = \frac{1}{N_S^3} \sum_{\mathbf{n}} \frac{1}{d_R} \mathrm{tr} \left[\prod_{n_4=0}^{N_T-1} U_4^R(n) \right] \qquad U_\mu^R(n) \approx e^{iA_\mu^R[(n+1/2)a]}$$

We implement triplet and octet representations R ($d_R = 3$, 8).

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"Physical interpretation"

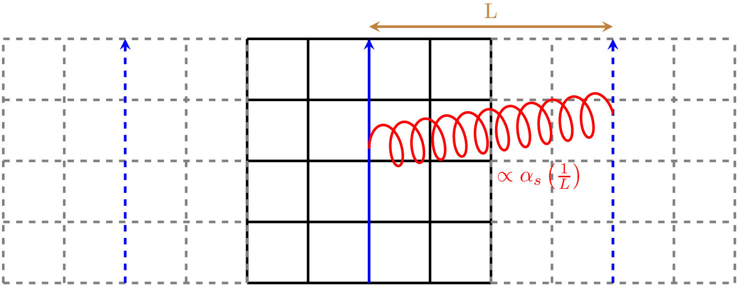

Figure: *Self-interactions with replicas producing $1/L = 1/(aN_S)$ Coulomb terms.*

$$\delta m^{(R)}(N_S) \propto \int_{1/(aN_S)}^{1/a} dk\, \alpha(k) \sim \frac{1}{a}\sum_n c_n \alpha^{n+1}\left(a^{-1}\right) - \frac{1}{aN_S}\sum_n c_n \alpha^{n+1}\left((aN_S)^{-1}\right) ,$$

$$c_n \simeq N_m \left(\frac{\beta_0}{2\pi}\right)^n n! , \qquad f_n^{(i)}(N_S) \simeq N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{n!}{i!} .$$

Perturbative OPE (Zimmermann) at finite volume ($N_S \to \infty$)

$$\delta m = \lim_{N_S\to\infty} \delta m(N_S) \quad c_n = \lim_{N_S\to\infty} c_n(N_S) \qquad \left(\lim_{n\to\infty} c_n^{(3,\rho)} = r_n(\nu)/\nu\right).$$

For large N_S, we write (OPE: $\frac{1}{a} \gg \frac{1}{N_S a}$)

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Taylor expansion of $\alpha\left((aN_S)^{-1}\right)$ in powers of $\alpha(a^{-1})$:

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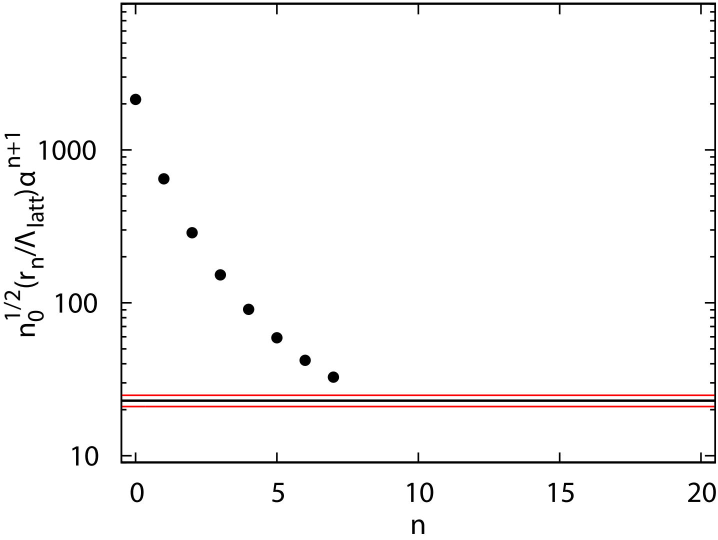
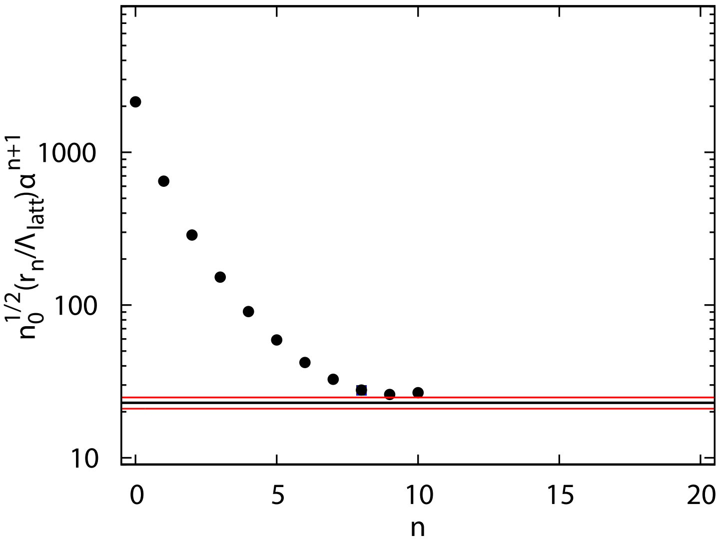

$n_0^{1/2}(r_n/\Lambda_{latt})\alpha^{n+1}$
1000
100
10
0
5
10
15
20
n



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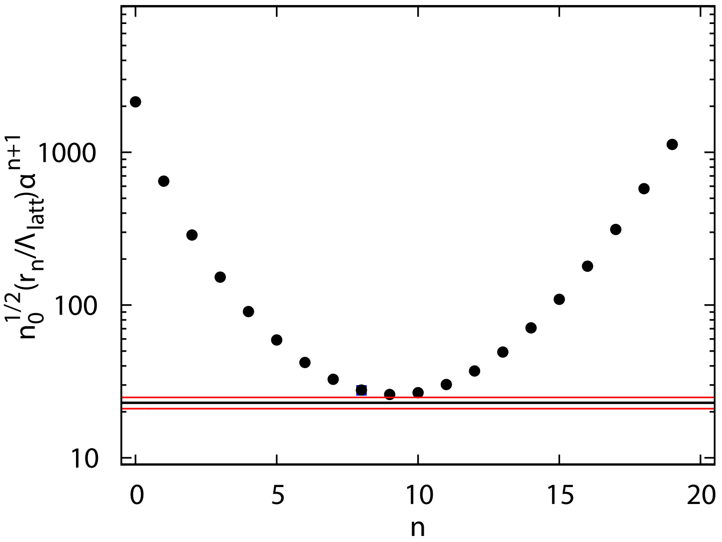

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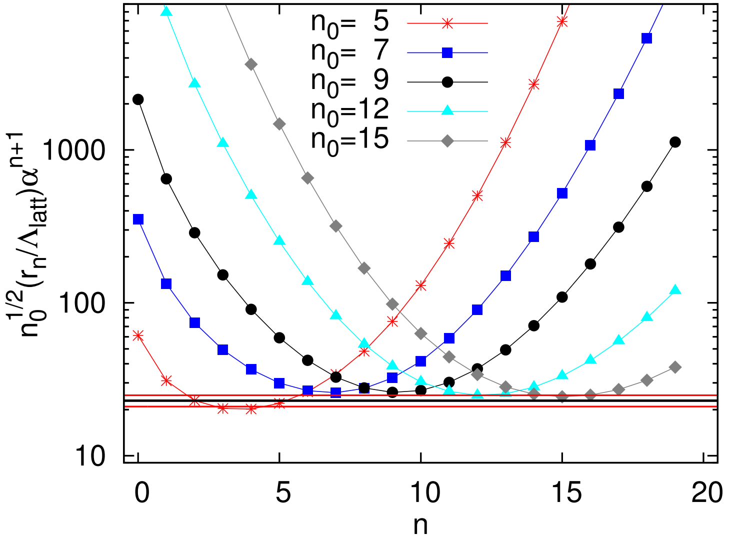

Figure: *c_n times $\sqrt{n_0}$, for five different values of the lattice scheme coupling constant α, ranging from $\alpha(\nu) \approx 0.096$ ($n_0 = 5$) to $\alpha(\nu) \approx 0.036$ ($n_0 = 15$). Bali, Bauer, AP, Torrero, 1303.3279.*

Ratios

$$
\frac{c_n^{(3,\rho)}}{c_{n-1}^{(3,\rho)}}\frac{1}{n} = \frac{c_n^{(8,\rho)}}{c_{n-1}^{(8,\rho)}}\frac{1}{n}
$$
$$
= \frac{\beta_0}{2\pi}\left\{1+\frac{b}{n}-\frac{bs_1}{n^2}+\frac{1}{n^3}\left[b^2 s_1^2+b(b-1)(s_1-2s_2)\right]+\mathcal{O}\left(\frac{1}{n^4}\right)\right\} \ .
$$

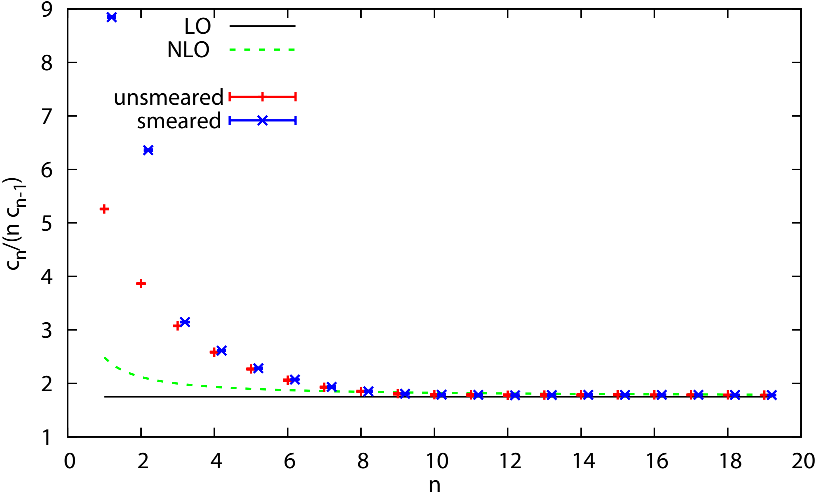

Figure: Ratios $c_n/(nc_{n-1})$ of the smeared (blue) and unsmeared (red) triplet static self-energy coefficients c_n in comparison to the theoretical prediction at different orders in the $1/n$ expansion.

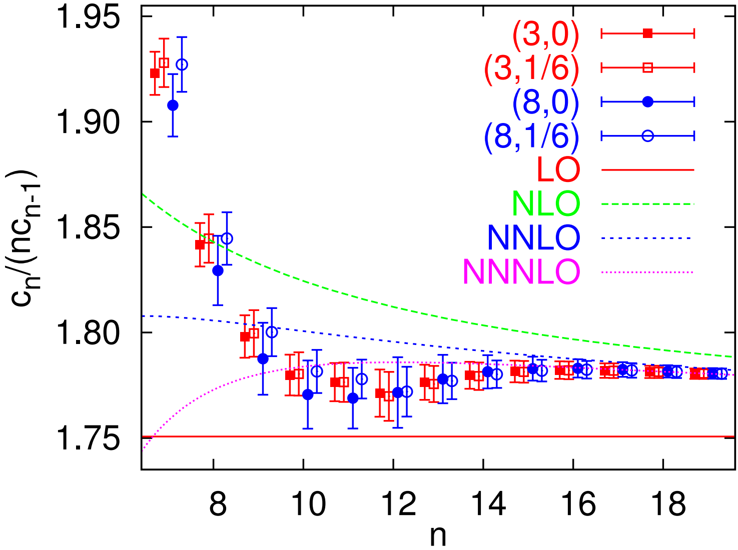

Figure: The ratios $c_n/(nc_{n-1})$ for the smeared and unsmeared, triplet and octet static self-energies, compared to the prediction for the LO, next-to-leading order (NLO), NNLO and NNNLO of the $1/n$ expansion.

N_m

$$c_n^{fitted} = N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)}\left(1+\frac{b}{(n+b)}c_1+\frac{b(b-1)}{(n+b)(n+b-1)}c_2+\cdots\right).$$

$$f_n^{fitted} = N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)}\left(1+\frac{b}{(n+b)}c_1+\frac{b(b-1)}{(n+b)(n+b-1)}c_2+\cdots\right).$$

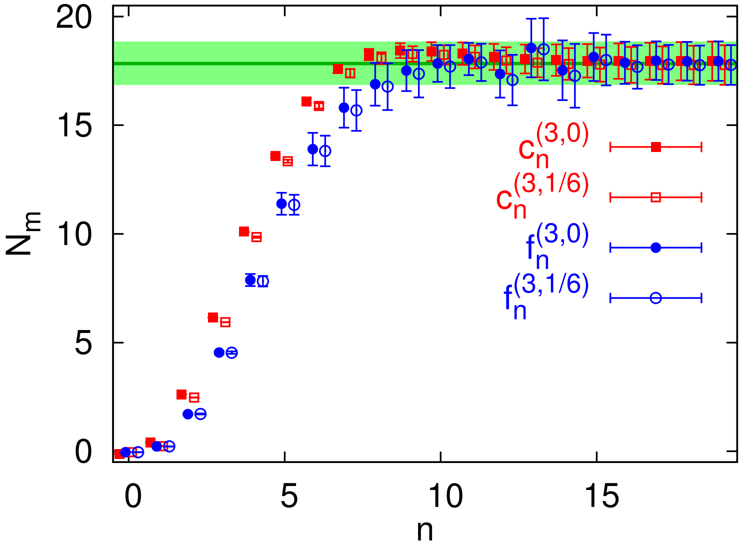

Figure: N_m^{latt} determined via r_n truncated at NNNLO, from the coefficients $c_n^{(3,0)}$, $c_n^{(3,1/6)}$, $f_n^{(3,0)}$ and $f_n^{(3,1/6)}$. The horizontal band is our final result: $N_m = 17.9(1.0)$

From lattice to $\overline{\text{MS}}$ scheme

$$\alpha_{\overline{\text{MS}}}(\mu) = \alpha_{\text{latt}}(\mu)\left(1 + d_1\alpha_{\text{latt}}(\mu) + d_2\alpha^2_{\text{latt}}(\mu) + d_3\alpha^3_{\text{latt}}(\mu) + \mathcal{O}(\alpha^4_{\text{latt}})\right) \,,$$

$$N^{\overline{\text{MS}}}_{m,m_{\tilde g}} = N^{\text{latt}}_{m,m_{\tilde g}}\Lambda_{\text{latt}}/\Lambda_{\overline{\text{MS}}} \,, \quad \text{where} \quad \Lambda_{\overline{\text{MS}}} = e^{\frac{2\pi d_1}{\beta_0}}\Lambda_{\text{latt}} \approx 28.809338139488\,\Lambda_{\text{latt}} \,.$$

This yields the numerical values

$$N^{\overline{\text{MS}}}_m = 0.620(35) \,, \quad C_F/C_A\, N^{\overline{\text{MS}}}_{m_{\tilde g}} = -C_F/C_A\, N^{\overline{\text{MS}}}_\Lambda = 0.610(41) \,.$$

$\sim$ 20 standard deviations from zero!
From $N^{\overline{\text{MS}}}_m = 0.600(29)$ (Ayala, Cvetic, AP). Combined $N^{\overline{\text{MS}}}_m = 0.608(22)$.

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Plaquette (Bali, Bauer, AP: 1401.7999, 1403.6477)

$$\langle P\rangle_{\rm pert}(N) \equiv \frac{1}{Z}\int [dU_{x,\mu}]\, e^{-S[U]} P[U]\Big|_{\rm NSPT} = \sum_{n\geq 0} p_n(N)\alpha^{n+1}$$

Perturbative OPE

$$\frac{1}{a} \gg \frac{1}{Na} \rightarrow \langle P\rangle_{\rm pert}(N) = P_{\rm pert}(\alpha)\langle 1\rangle + \frac{\pi^2}{36} C_{\rm G}(\alpha)\, a^4\langle G^2\rangle_{\rm soft} + \mathcal{O}\left(\frac{1}{N^6}\right),$$

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$$\langle P\rangle_{\rm pert}(N) \equiv \frac{1}{Z}\int [dU_{x,\mu}]\, e^{-S[U]} P[U]\Big|_{\rm NSPT} = \sum_{n\geq 0} p_n(N)\alpha^{n+1}$$

Perturbative OPE

$$\frac{1}{a} \gg \frac{1}{Na} \to \langle P\rangle_{\rm pert}(N) = P_{\rm pert}(\alpha)\langle 1\rangle + \frac{\pi^2}{36} C_{\rm G}(\alpha)\, a^4 \langle G^2\rangle_{\rm soft} + \mathcal{O}\left(\frac{1}{N^6}\right),$$

where

$$P_{\rm pert} = \sum_{n\geq 0} p_n\alpha^{n+1}, C_{\rm G} = 1+\sum_{k\geq 0} c_k\alpha^{k+1}, \frac{\pi^2}{36} a^4\langle G^2\rangle_{\rm soft} = -\frac{1}{N^4}\sum_{n\geq 0} f_n\alpha^{n+1}((Na)^{-1})$$

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$$= \sum_{n\geq 0} p_n\alpha^{n+1} - \frac{1}{N^4}\left(1+\sum_{k\geq 0} c_k\alpha^{k+1}(a^{-1})\right)\times \sum_{n\geq 0} f_n\alpha^{n+1}((Na)^{-1}) + \mathcal{O}\left(\frac{1}{N^6}\right),$$

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$$d = 1 (n_0 \sim 7) \longrightarrow d = 4 (n_0 \sim 28)$$

$$N + 1 = 35$$

(before Di Renzo et al. N+1=8; Horsley et al. N+1=20)
Renormalon expectations:

$$p_n^{\rm latt} \overset{n\to\infty}{=} N_P^{\rm latt}\left(\frac{\beta_0}{2\pi d}\right)^n \frac{\Gamma(n+1+db)}{\Gamma(1+db)}\left\{1 + \frac{20.09}{n+db} + \frac{505\pm 33}{(n+db)^2} + \mathcal{O}\left(\frac{1}{n^3}\right)\right\}.$$

$$\frac{p_n}{np_{n-1}} = \frac{\beta_0}{2\pi d}\left\{1 + \frac{db}{n} + \frac{db(1-ds_1)}{n^2} + \frac{\#}{n^3} + \mathcal{O}\left(\frac{1}{n^4}\right)\right\}.$$

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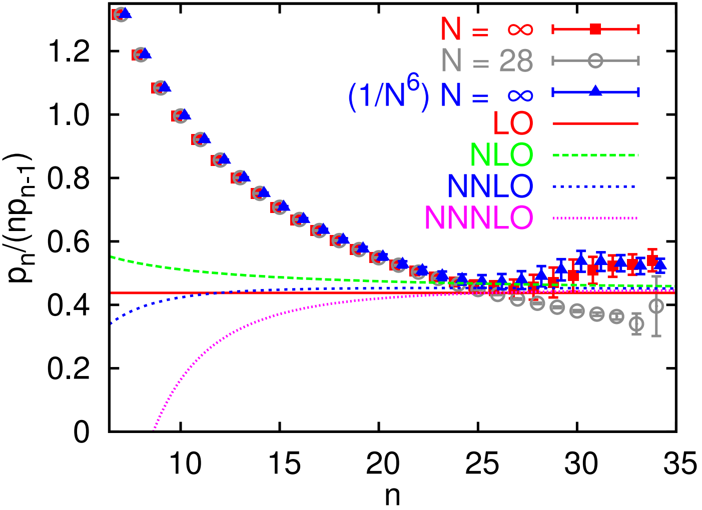


Figure: Ratios $p_n/(np_{n-1})$ of the plaquette coefficients p_n ($N=\infty$, $N=28$) in comparison to the theoretical prediction at different orders in the $1/n$ expansion.

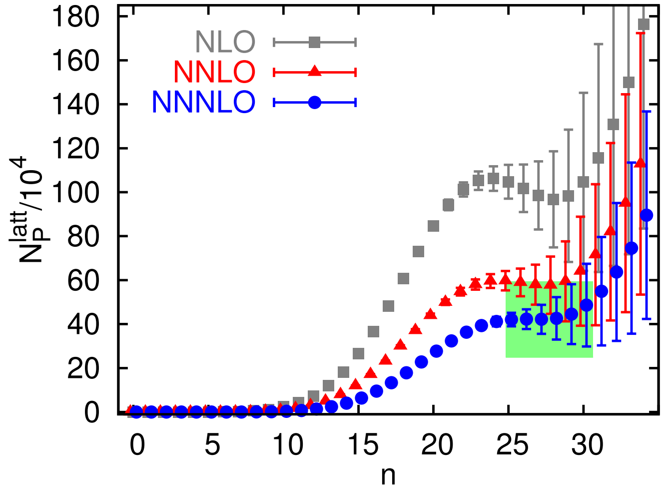
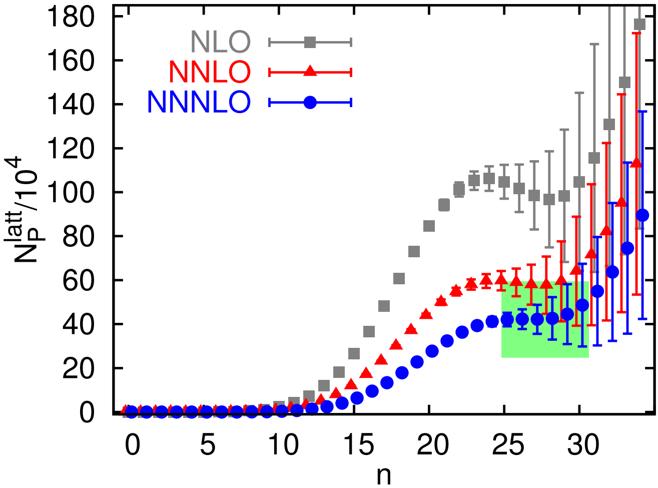


Figure: N_P, determined from the coefficients p_n truncated at NLO, NNLO and NNNLO. The green box marks our final result.

$$N_P^{\overline{\text{MS}}} = 0.61(25) \qquad N_G^{\overline{\text{MS}}} = \frac{36}{\pi^2} N_P^{\overline{\text{MS}}} = 2.24(92)\,.$$



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Uncertainty of the sum due to the truncation

$$\delta S_P = \sqrt{n_0}\, p_{n_0} \alpha^{n_0+1} \approx \frac{(2\pi)^{3/2} d^{1+db}}{2^{db}\beta_0\,\Gamma(1+db)} N_P(\Lambda a)^4 \approx 12.06\, N_P(\Lambda a)^4 \,.$$

This object is scheme- and scale-independent (to $1/n$-precision)

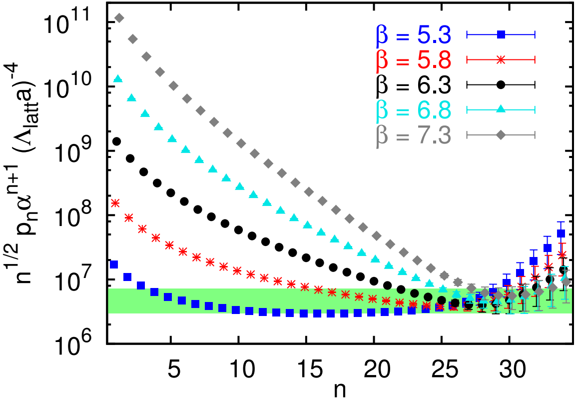

Figure: $\sqrt{n}\, p_n \alpha^{n+1}/(\Lambda_{\rm latt} a)^4$, versus n for $\beta = 5.3, 5.8, 6.3, 6.8$ and 7.3. The green band is the theoretical expectation $12.06\, N_P = 5.1(2.1)\times 10^6$.

$$\sqrt{n_0}\frac{|r_{n_0}|}{\Lambda_{\rm latt}}\alpha^{n_0+1}(\nu) = \frac{2^{3/2-b}\pi^{3/2}}{\beta_0\Gamma(1+b)}|N_m| \approx 1.206\,|N_m|\,,$$

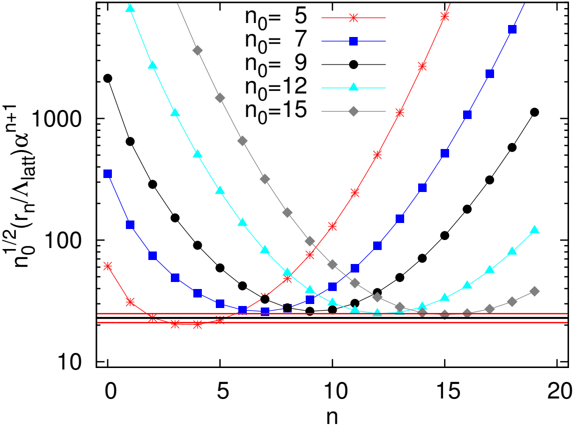

Figure: c_n times $\sqrt{n_0}$, for five different values of the lattice scheme coupling constant α, ranging from $\alpha(\nu) \approx 0.096$ ($n_0 = 5$) to $\alpha(\nu) \approx 0.036$ ($n_0 = 15$).

$$\delta\langle G^2\rangle_{\rm NP} \simeq \frac{(2\pi)^{3/2} d^{1+db}}{2^{db}\beta_0\,\Gamma(1+db)} N_{\rm G}^{\overline{\rm MS}}\Big|_{n_l=0} \Lambda_{\overline{\rm MS}}^4 = 27(11)\,\Lambda_{\overline{\rm MS}}^4 \sim 0.087\,{\rm GeV}^4\,.$$

$$\langle G^2\rangle = 3.18(29) r_0^{-4} = 24.2(8.0)\Lambda_{\overline{\rm MS}}^4 \simeq 0.077\,{\rm GeV}^4\,.$$

CONCLUSIONS

Renormalons go beyond large-β_0 analysis: $\rightarrow$ (NP)OPE

Strong evidence of renormalon dominance in heavy quark physics from $\mathcal{O}(\alpha^{3/4})$ $\overline{\rm MS}$-like computations: Pole mass, static potential, $\cdots$

$$N_m^{\overline{\rm MS}}(n_l=0)=0.600(29)\,,\quad N_m^{\overline{\rm MS}}(n_l=3)=0.563(26)\,.$$

Lattice: For the first time it was possible to follow the factorial growth of the coefficients over many orders, from around α^9 up to α^{20}, vastly increasing the credibility of the prediction.

$$N_m^{\overline{\rm MS}}(n_l=0)=0.620(35)\,,\quad C_F/C_A\,N_\Lambda^{\overline{\rm MS}}(n_l=0)=-0.610(41)\,.$$

Two independent determinations with very different systematics.
We have (numerically) proven, beyond any reasonable doubt ($\sim$ 20 standard deviations!), the existence of the renormalon in QCD.

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Dimension two condensates: artifacts of incomplete subtractions

- unquantifible error due to the simplified parameterization of higher order perturbation theory
- short distance effect $\rightarrow$ process dependent

FUTURE:

n_f dependence

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Beyond perturbation theory (at last...)

$$\langle P \rangle_{\rm pert} = \frac{1}{Z}\int [dU_{x,\mu}]\, e^{-S[U]} P[U]\Big|_{\rm NSPT} = P_{\rm pert}(\alpha)\langle 1 \rangle + \frac{\pi^2}{36} C_{\rm G}(\alpha)\, a^4 \langle O_{\rm G}\rangle_{\rm soft} + \mathcal{O}\left(a^6\right) .$$

$$\frac{1}{a} \gg \frac{1}{Na}$$

$$\langle P \rangle_{\rm MC} = \frac{1}{Z}\int [dU_{x,\mu}]\, e^{-S[U]} P[U]\Big|_{\rm MC} = P_{\rm pert}(\alpha)\langle 1 \rangle + \frac{\pi^2}{36} C_{\rm G}(\alpha)\, a^4 \langle G^2\rangle_{\rm MC} + \mathcal{O}\left(a^6\right) .$$

$$\frac{1}{a} \gg \frac{1}{Na} \gg \Lambda_{\rm QCD} \quad \rightarrow \quad \langle G^2\rangle_{\rm MC} = \langle G^2\rangle_{\rm soft}\left[1 + \mathcal{O}(\Lambda_{\rm QCD}^2 (Na)^2)\right]$$

$$\frac{1}{a} \gg \Lambda_{\rm QCD} \gg \frac{1}{Na} \quad \rightarrow \quad \langle G^2\rangle_{\rm MC} = \langle G^2\rangle_{\rm NP}\left[1 + \mathcal{O}\left(\frac{1}{\Lambda_{\rm QCD}^2 (Na)^2}\right)\right],$$

where $\langle G^2\rangle_{\rm NP} \sim \Lambda_{\rm QCD}^4$ is the NP gluon condensate (Vainshtein, Zakharov, Shifman).

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In this limit non-perturbative effects can be computed at weak coupling (still far from trivial: see resurgence analysis in 1+1 dimensions).

Observations:

- In this limit the gluon condensate renormalon is not produced by non-perturbative effects.
- The resummation of all $(\Lambda^2_{\rm QCD}(Na)^2)^n$ effects remains to be done to reach the scaling region at infinite volume: $\frac{1}{a} \gg \Lambda_{\rm QCD} \gg \frac{1}{Na}$, i.e. $\langle G^2\rangle_{\rm NP}$.

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Determination of the gluon condensate: $\frac{1}{a} \gg \Lambda_{\rm QCD} \gg \frac{1}{Na}$

$$\langle G^2 \rangle_{\rm NP} = \frac{36 C_{\rm G}^{-1}(\alpha)}{\pi^2 a^4(\alpha)} \left[\langle P \rangle_{\rm MC}(\alpha) - S_P(\alpha) \right] + \mathcal{O}(a^2 \Lambda_{\rm QCD}^2) \, .$$

$$S_P(\alpha) \equiv S_{n_0}(\alpha) \, , \quad \text{where} \quad S_n(\alpha) = \sum_{j=0}^{n} p_j \alpha^{j+1} \, .$$

$n_0 \equiv n_0(\alpha)$ is the order for which $p_{n_0} \alpha^{n_0+1}$ is minimal.

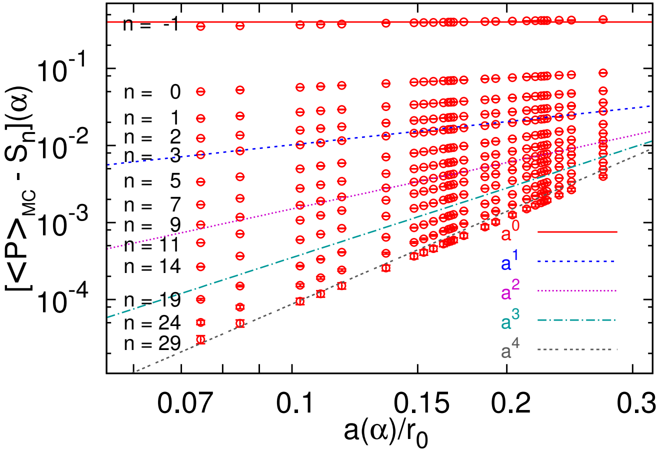

Figure: $\langle P\rangle_{\rm MC}(\alpha) - S_n(\alpha)$ between MC data and sums truncated at orders α^{n+1} ($S_{-1} = 0$) vs. $a(\alpha)/r_0$. The lines $\propto a^j$ are drawn to guide the eye.

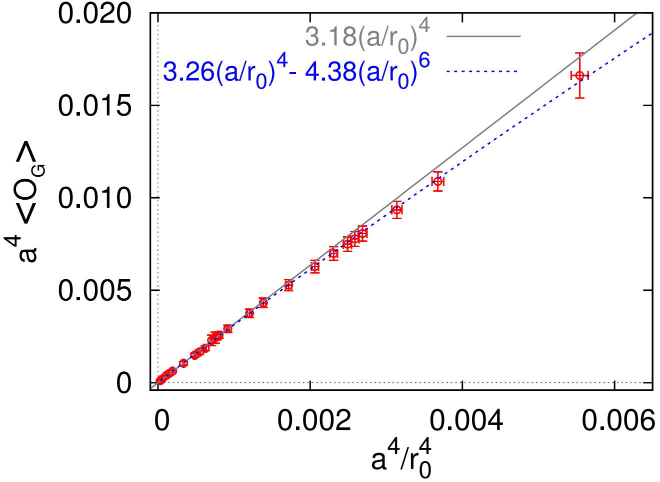

Figure: $\langle P \rangle_{\mathrm{MC}}(\alpha) - S_P(\alpha)$. The linear fit is to $a^4 < 0.0013\, r_0^4$ points only.

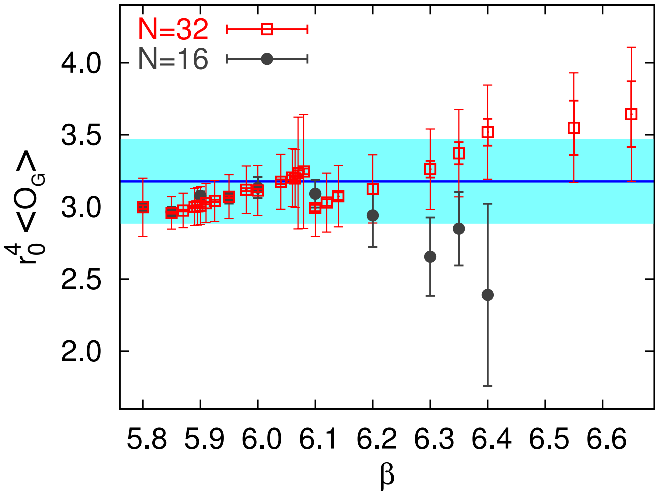

Figure: $\langle G^2 \rangle$ evaluated using the $N = 16$ and $N = 32$ MC data of Boyd et al. The error band is our prediction for $\langle G^2 \rangle$.

$$\langle G^2 \rangle = 3.18(29) r_0^{-4} = 24.2(8.0) \Lambda_{\overline{\text{MS}}}^4 \simeq 0.077\,\text{GeV}^4 \,.$$

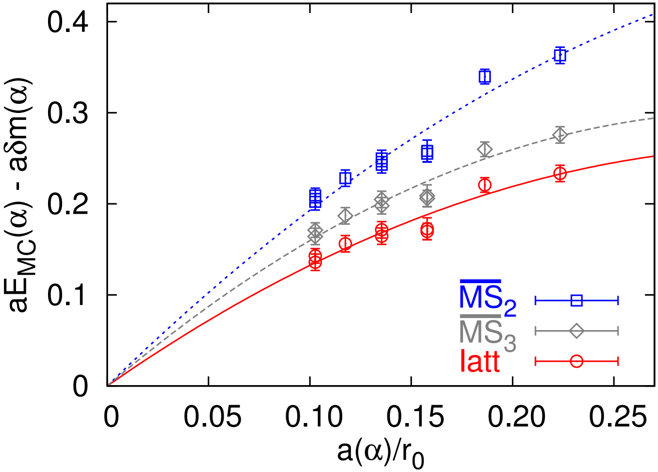

Figure: $aE_{\rm MC} - a\delta m$ vs. a/r_0. The expansion of $a\delta m$ was also converted into the $\overline{\rm MS}$ scheme at two ($\overline{\rm MS}_2$) and three ($\overline{\rm MS}_3$) loops. The curves are fits to $\overline{\Lambda}a + ca^2$.

Determination of N_m

$\nu \sim m$

Large β_0 analysis

$$m\left(\frac{\nu}{m}\right)^{2u} \simeq \nu\{1+(2u-1)\ln\frac{\nu}{m}+\cdots\}.$$

Therefore, the underlying assumption is that we are in a regime where (besides $2u-1 \ll 1$)

$$(2u-1)\ln\frac{\nu}{m} \ll 1\,.$$

$$\boxed{N_m = \frac{r_n}{(r_n^{asym}/N_m)}}$$

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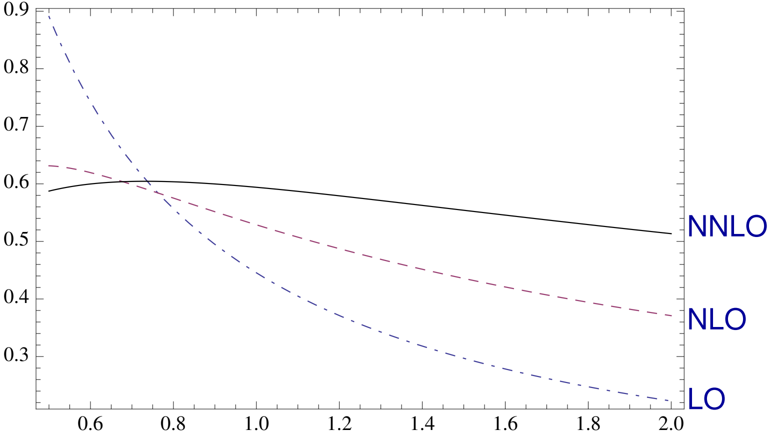

Figure: N_m for $n_l = 3$, as a function of $x \equiv \mu/m_b$, obtained from r_n/r_n^{asym} with r_n^{asym} truncated at $\mathcal{O}(1/n^3)$. We name the different lines as NLO (dashed-dotted), NLO (dashed) and NNLO (solid) for $n = 0, 1, 2$, respectively.

The static potential

$$V(r;\nu_{us}) = \sum_{n=0}^{\infty} V_n \alpha_s^{n+1},$$

$2m_{\rm OS} + V$ can be understood as an observable up to $O(r^2\Lambda^3_{\rm QCD}, \Lambda^2_{\rm QCD}/m)$ contributions $\to 2N_m + N_V = 0$ and

$$V_n^{asym} = N_V\,\nu\left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)}\left(1+\frac{b}{(n+b)}c_1+\frac{b(b-1)}{(n+b)(n+b-1)}c_2+\cdots\right)$$

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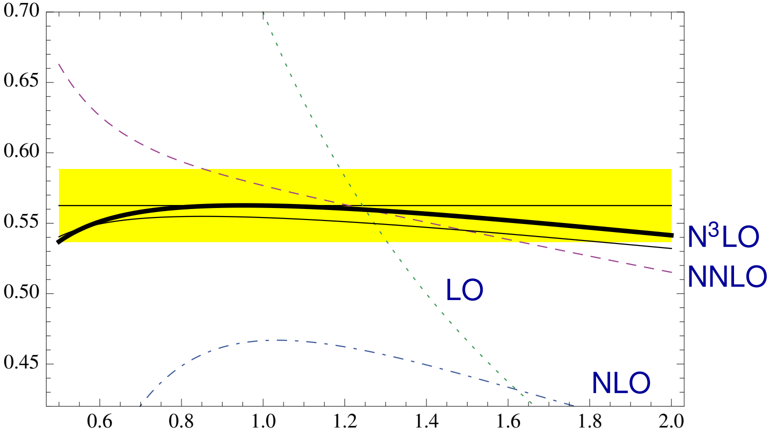

Figure: $-N_V/2 = N_m$ for $n_l = 3$, as a function of $x \equiv \nu r$, obtained from $-(N_V/2)v_n/v_n^{asym}$. v_n^{asym} is truncated at $\mathcal{O}(1/n^3)$.

$$\boxed{N_m(n_l = 0) = 0.600(29)}, \qquad \boxed{N_m(n_l = 3) = 0.563(26)}.$$

$\sim$ 20 standard deviations from zero!

Conformal window: n_l dependence

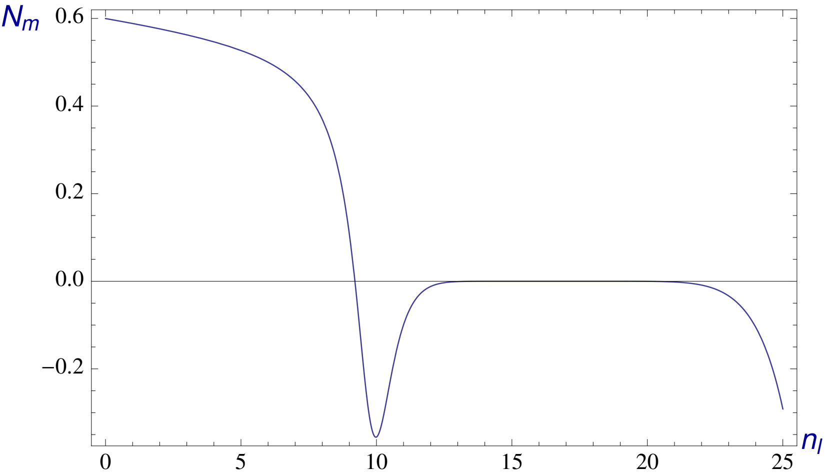

Figure: $N_m(x=1)$ *obtained from* $-(N_V/2)v_3/v_3^{asym}$ *as a function of* n_l.

First numerical evidence of the disappearance of the renormalon in the conformal window.

	$c_n^{(3,0)}$	$c_n^{(3,1/6)}$	$c_n^{(8,0)}C_F/C_A$	$c_n^{(8,1/6)}C_F/C_A$
c_0	2.117274357	0.72181(99)	2.117274357	0.72181(99)
c_1	11.136(11)	6.385(10)	11.140(12)	6.387(10)
$c_2/10$	8.610(13)	8.124(12)	8.587(14)	8.129(12)
$c_3/10^2$	7.945(16)	7.670(13)	7.917(20)	7.682(15)
$c_4/10^3$	8.215(34)	8.017(33)	8.197(42)	8.017(36)
$c_5/10^4$	9.322(59)	9.160(59)	9.295(76)	9.139(64)
$c_6/10^6$	1.153(11)	1.138(11)	1.144(13)	1.134(12)
$c_7/10^7$	1.558(21)	1.541(22)	1.533(25)	1.535(22)
$c_8/10^8$	2.304(43)	2.284(45)	2.254(51)	2.275(45)
$c_9/10^9$	3.747(95)	3.717(97)	3.64(11)	3.703(98)
$c_{10}/10^{10}$	6.70(22)	6.65(22)	6.49(25)	6.63(22)
$c_{11}/10^{12}$	1.316(52)	1.306(53)	1.269(59)	1.303(53)
$c_{12}/10^{13}$	2.81(13)	2.79(13)	2.71(14)	2.78(13)
$c_{13}/10^{14}$	6.51(35)	6.46(35)	6.29(37)	6.45(35)
$c_{14}/10^{16}$	1.628(96)	1.613(97)	1.57(10)	1.614(97)
$c_{15}/10^{17}$	4.36(28)	4.32(28)	4.22(29)	4.33(28)
$c_{16}/10^{19}$	1.247(86)	1.235(86)	1.206(89)	1.236(86)
$c_{17}/10^{20}$	3.78(28)	3.75(28)	3.66(28)	3.75(28)
$c_{18}/10^{22}$	1.215(93)	1.204(94)	1.176(95)	1.205(94)
$c_{19}/10^{23}$	4.12(33)	4.08(33)	3.99(34)	4.08(33)

	$f_n^{(3,0)}$	$f_n^{(3,1/6)}$	$f_n^{(8,0)} C_F/C_A$	$f_n^{(8,1/6)} C_F/C_A$
f_0	0.7696256328	0.7810(59)	0.7696256328	0.7810(69)
f_1	6.075(78)	6.046(58)	6.124(87)	6.063(68)
$f_2/10$	5.628(91)	5.644(62)	5.60(11)	5.691(78)
$f_3/10^2$	5.87(11)	5.858(76)	6.00(18)	5.946(91)
$f_4/10^3$	6.33(22)	6.29(17)	6.57(40)	6.26(23)
$f_5/10^4$	7.73(35)	7.71(26)	7.67(66)	7.78(42)
$f_6/10^5$	9.86(53)	9.80(42)	9.68(99)	9.79(69)
$f_7/10^7$	1.388(81)	1.378(71)	1.35(15)	1.38(11)
$f_8/10^8$	2.12(12)	2.11(12)	2.06(22)	2.10(17)
$f_9/10^9$	3.54(20)	3.52(20)	3.40(37)	3.51(27)
$f_{10}/10^{10}$	6.49(33)	6.44(34)	6.23(67)	6.44(43)
$f_{11}/10^{12}$	1.296(64)	1.286(66)	1.24(13)	1.286(74)
$f_{12}/10^{13}$	2.68(19)	2.64(18)	2.65(33)	2.65(21)
$f_{13}/10^{14}$	6.70(54)	6.68(52)	6.36(90)	6.66(57)
$f_{14}/10^{16}$	1.58(14)	1.56(14)	1.55(22)	1.57(15)
$f_{15}/10^{17}$	4.41(34)	4.37(33)	4.24(47)	4.37(35)
$f_{16}/10^{19}$	1.241(92)	1.230(91)	1.20(11)	1.231(94)
$f_{17}/10^{20}$	3.79(28)	3.75(28)	3.67(30)	3.76(28)
$f_{18}/10^{22}$	1.215(94)	1.204(94)	1.176(97)	1.205(94)
$f_{19}/10^{23}$	4.12(33)	4.08(33)	3.99(34)	4.08(33)

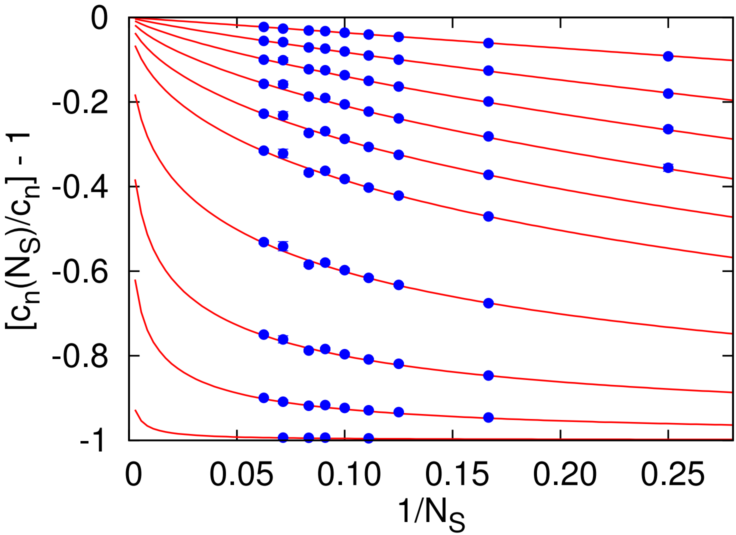

Figure: $c_n^{(3,0)}(N_S)/c_n^{(3,0)} - 1$ *for* $n \in \{0,1,2,3,4,5,7,9,11,15\}$ *(top to bottom). For each value of* N_S *we have plotted the data point with the maximum value of* N_T*. The curves represent the global fit.* $-(1/N_S)f_{0,\mathrm{DLPT}}^{(3,0)}/c_{0,\mathrm{DLPT}}^{(3,0)}$ *is shown for* $n = 0$.

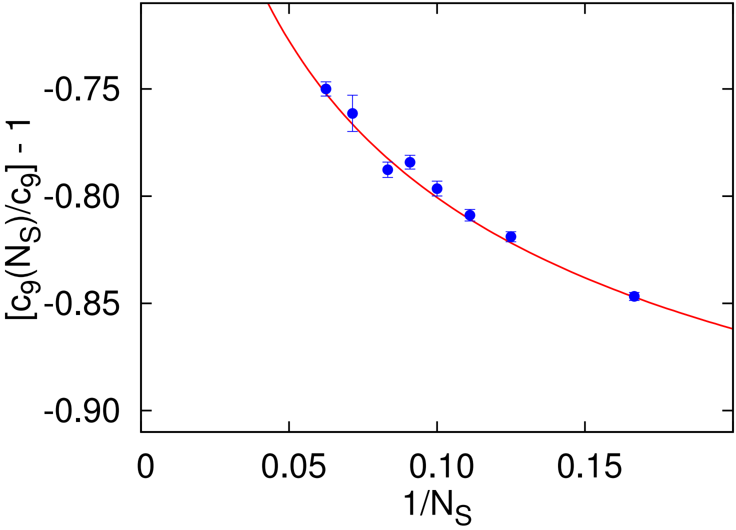

Figure: *Zoom of previous Figure for $n = 9$.*