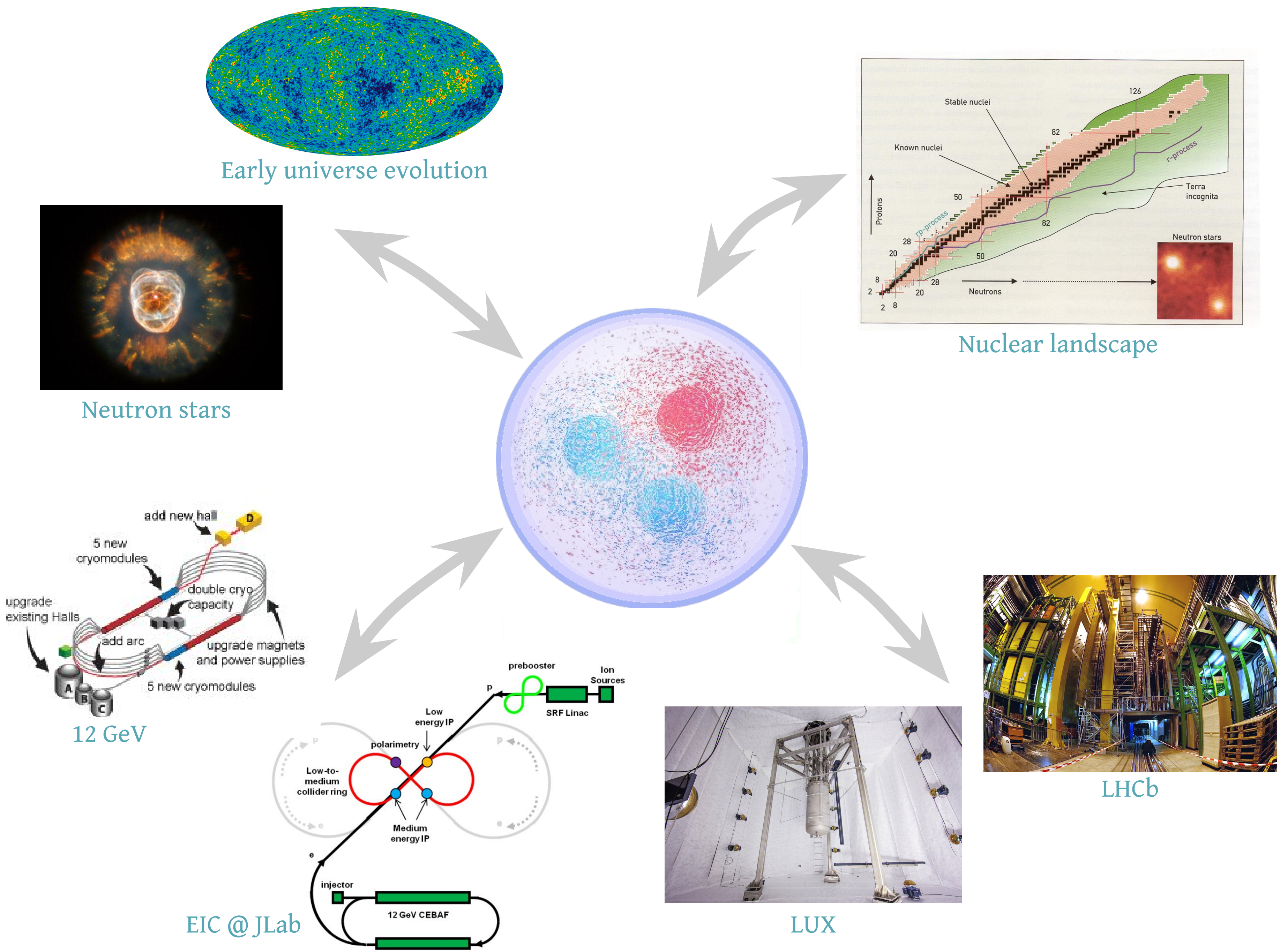
Operator product expansion with gradient flow

Christopher Monahan
New High Energy Theory Center
Rutgers, The State University of New Jersey

with Kostas Orginos


Early universe evolution
Stable nuclei
Known nuclei
126
82
50
28
20
8
2
rp-process
r-process
Terra incognita
Protons
Neutrons
Neutron stars
Nuclear landscape
Neutron stars
add new hall
D
5 new cryomodules
double cryo capacity
upgrade existing Halls
add arc
upgrade magnets and power supplies
5 new cryomodules
A
B
C
12 GeV
prebooster
Ion Sources
SRF Linac
p
Low energy IP
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Low-to-medium collider ring
Medium energy IP
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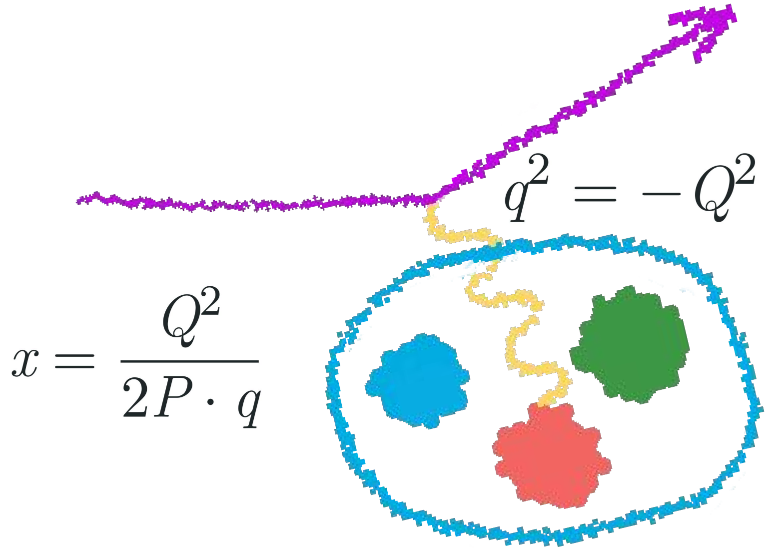
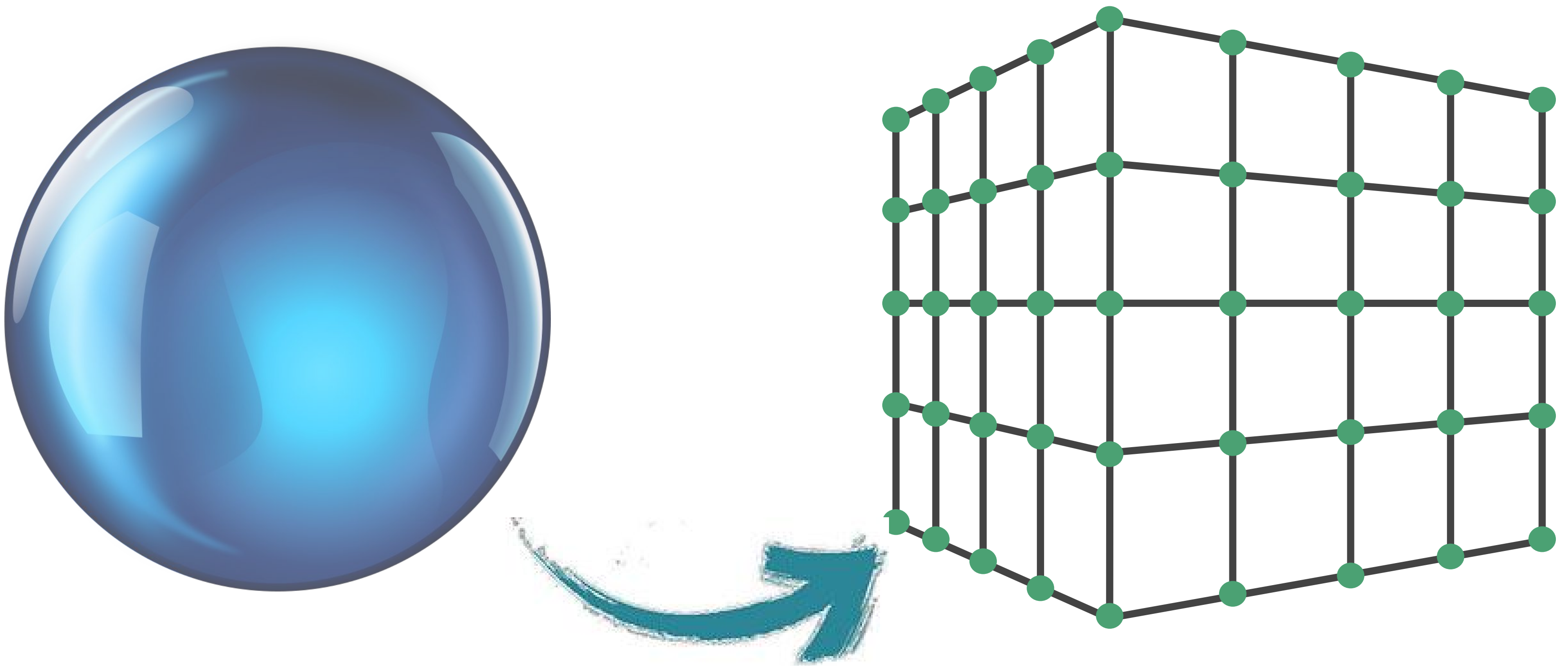
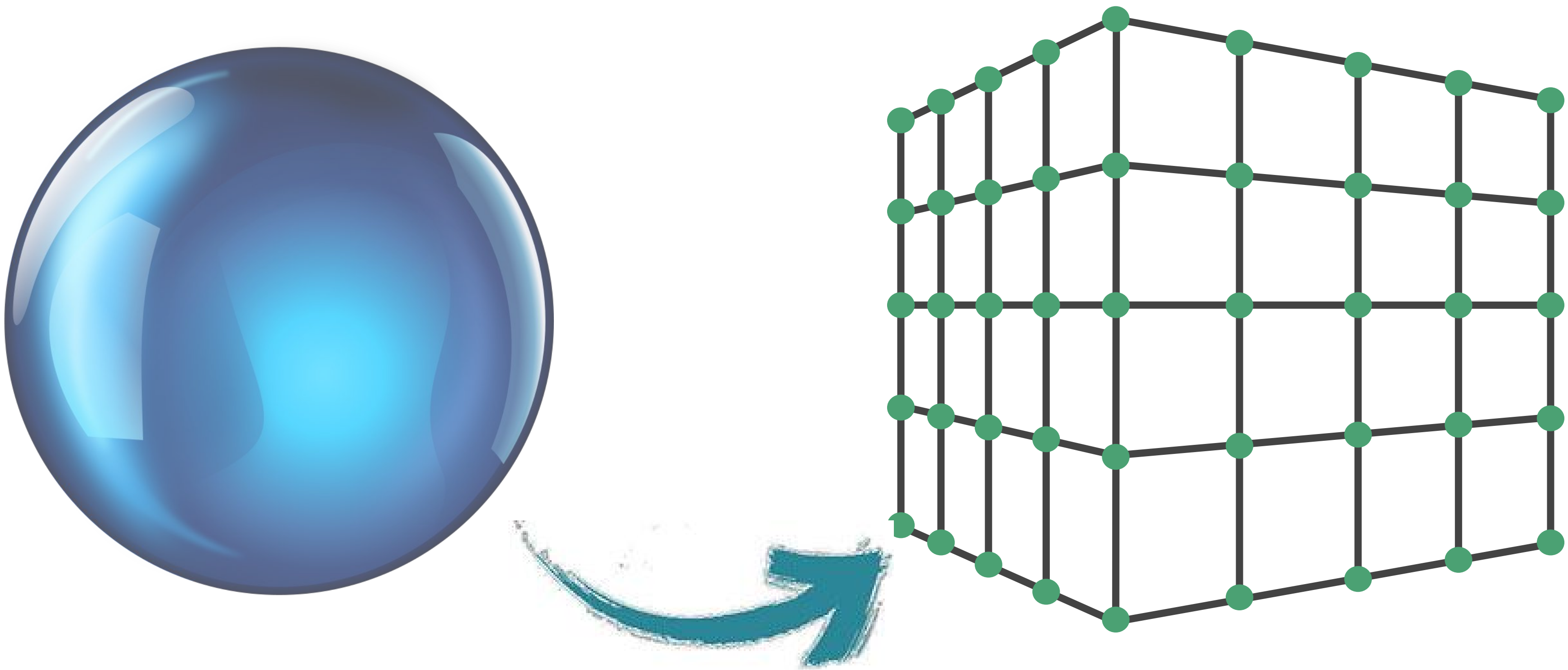
injector
12 GeV CEBAF
EIC @ JLab
LUX
LHCb



$$\langle x^n \rangle_{f_{q/N}} = \int_{-1}^{1} \mathrm{d}x \, x^n f_{q/N}(x)$$

$$2\langle x^n \rangle_{f_{q/N}} P_{\mu_1} \cdots P_{\mu_n} = \frac{1}{2} \langle N(P) | \overline{\psi} \gamma_{\{\mu_1} \overleftrightarrow{D}_{\mu_2} \cdots \overleftrightarrow{D}_{\mu_n\}} \psi | N(P) \rangle$$



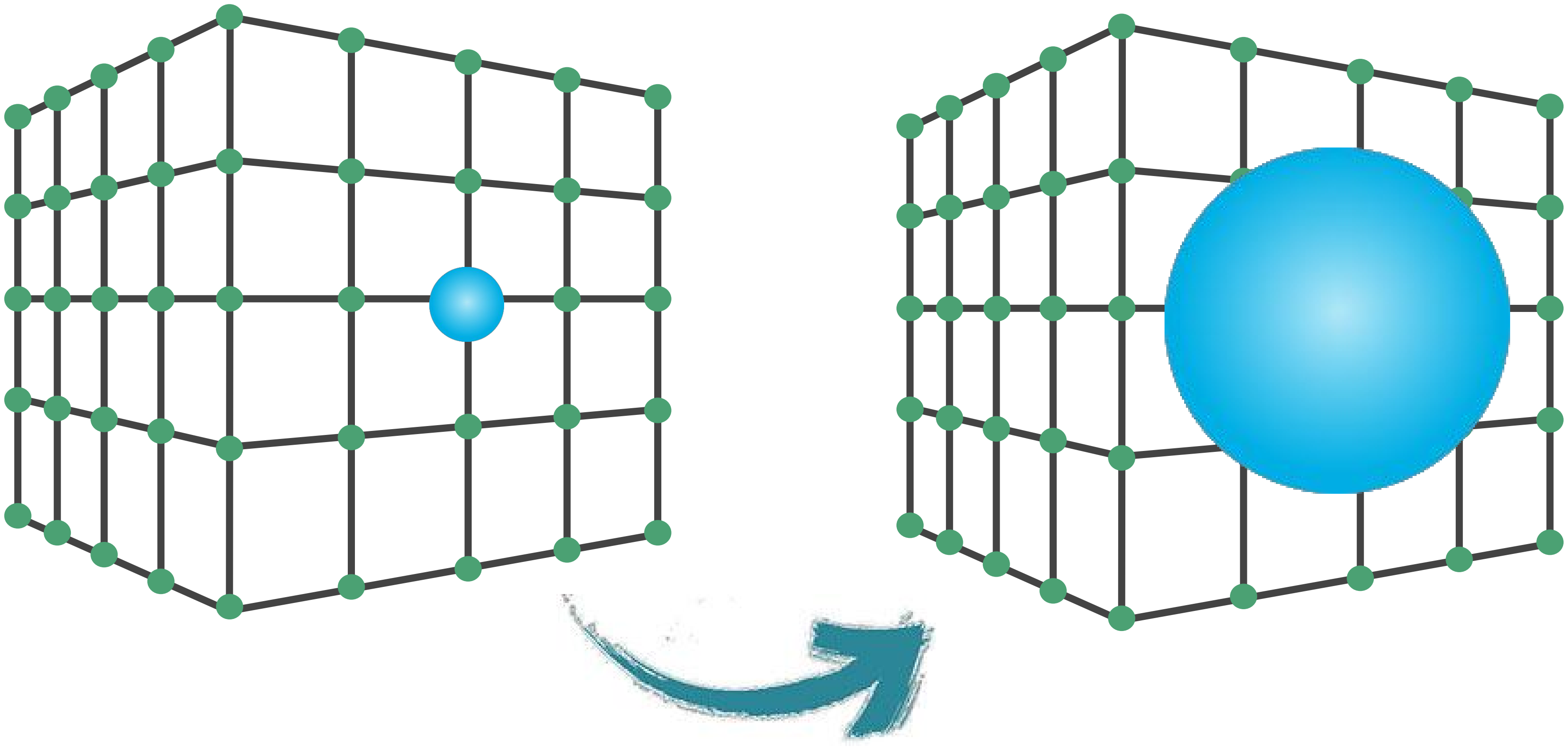


$$2\langle x^n \rangle_{f_{q/N}} P_{\mu_1}\cdots P_{\mu_n} = \frac{1}{2}\langle N(P)|\overline{\psi}\gamma_{\{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n\}}\psi|N(P)\rangle$$

$$\overline{\psi}\gamma_4\gamma_5\overleftrightarrow{D}_4\overleftrightarrow{D}_4\psi \sim \frac{1}{a^2}\overline{\psi}\gamma_4\gamma_5\psi$$

Power-divergent mixing restricts lattice calculations to first four moments

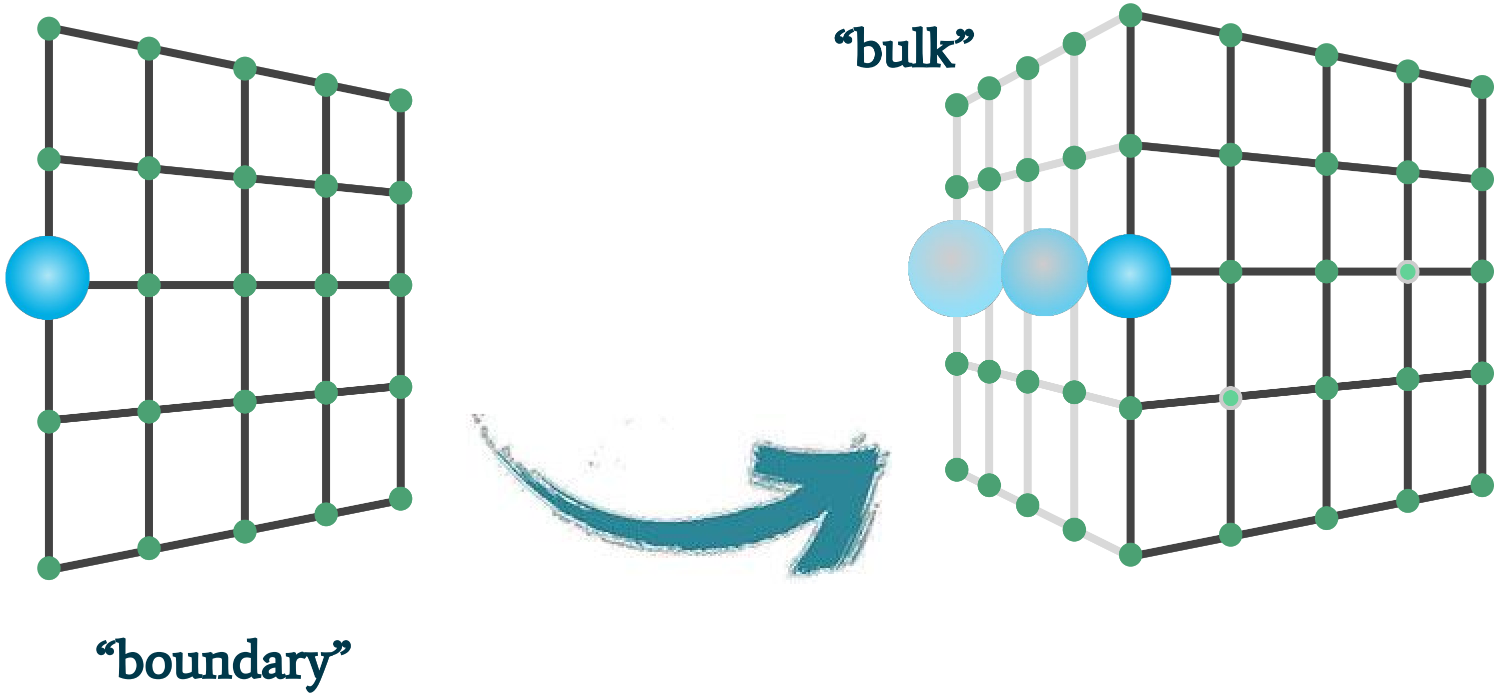
Detmold *et al.*, Eur. Phys. J. C 3 (2001) 1
Detmold *et al.*, Phys. Rev. D 68 (2001) 034025
Detmold *et al.*, Mod. Phys. Lett. A 18 (2003) 2681



Gradient flow: deterministic evolution in new parameter - flow time

Drives fields to minimise action - removes UV fluctuations

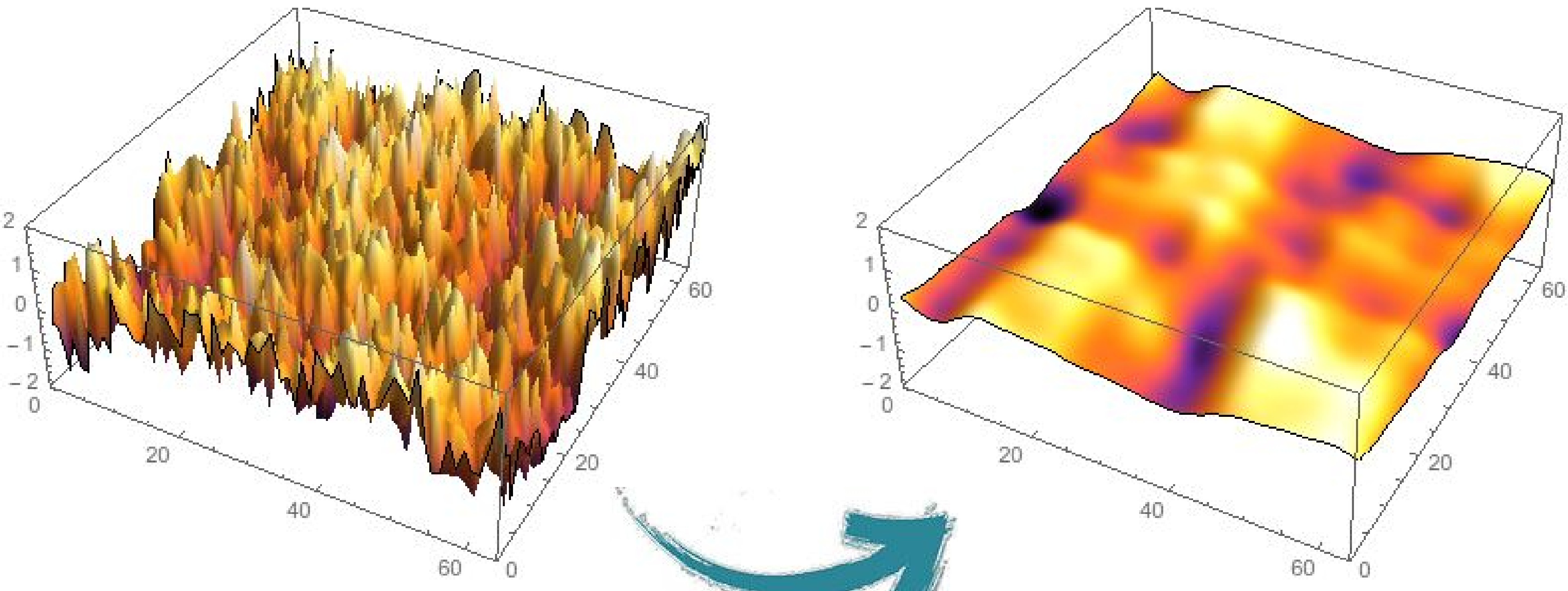
Narayanan & Neuberger, JHEP 0603 (2006) 064
Lüscher, Commun. Math. Phys. 293 (2010) 899



Gradient flow ensures **renormalised boundary theory remains finite**

Gradient flow: deterministic evolution in new parameter - flow time

Drives fields to minimise action - removes UV fluctuations



CJM, PoS(Lattice2015) 052

Gradient flow ensures **renormalised boundary theory remains finite**

Scalar field theory

$$\frac{\partial}{\partial\tau}\overline{\phi}(\tau,x) = \partial^2\overline{\phi}(\tau,x) \qquad \overline{\phi}(\tau\!=\!0,x) = \phi(x) \qquad \widetilde{\overline{\phi}}(\tau,p) = e^{-\tau p^2}\widetilde{\phi}(p)$$

CJM & K. Orginos, PRD 91 (2015) 074513

Exact solution possible with Dirichlet boundary conditions

$$\overline{\phi}(\tau,x) = \int \mathrm{d}^4 y \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{ip\cdot(x-y)} e^{-\tau p^2}\phi(y) = \frac{1}{16\pi^2\tau^2}\int \mathrm{d}^4 y\, e^{-(x-y)^2/(4\tau)}\phi(y)$$

Smearing radius $s_{\mathrm{rms}} = \sqrt{8\tau}$

Interactions occur at zero flow time (*i.e.* on the boundary)

Gradient flow in QCD

$$\frac{\partial}{\partial\tau}B_\mu(\tau,x) = D_\nu\Big(\partial_\nu B_\mu - \partial_\mu B_\nu + [B_\nu, B_\mu]\Big) \qquad D_\mu = \partial_\mu + [B_\mu, \cdot\,]$$

$$\frac{\partial}{\partial\tau}\chi(\tau,x) = D^F_\mu D^F_\mu \chi(\tau,x) \qquad D^F_\mu = \partial_\mu + B_\mu$$

Dirichlet boundary conditions

$$B_\mu(\tau=0,x) = A_\mu(x) \qquad \chi(\tau=0,x) = \psi(x)$$

Tree-level expansion

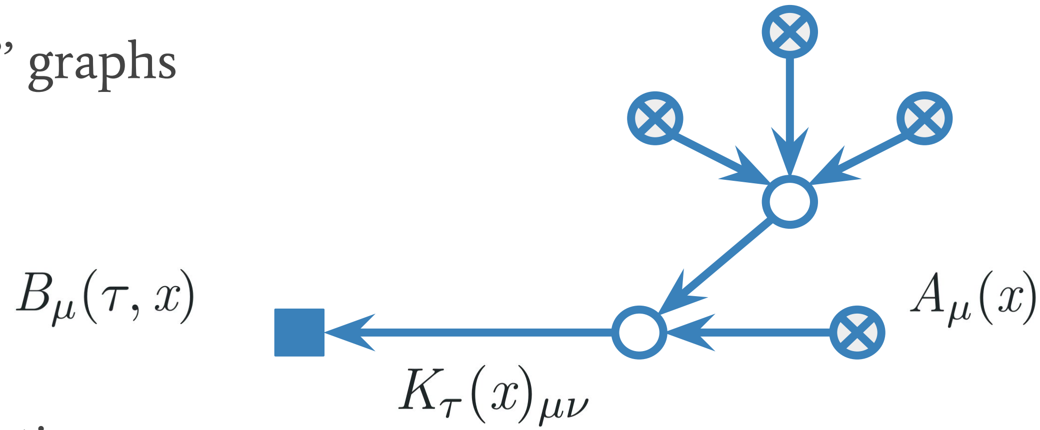
$$B_\mu(\tau,x) = \int \mathrm{d}^4y \Big\{ K_\tau(x-y)_{\mu\nu} A_\nu(y) + \int_0^\tau \mathrm{d}\sigma\, K_{\tau-\sigma}(x-y)_{\mu\nu} R_\nu(\sigma,y) \Big\}$$

"Flow propagator"

$$K_\tau(x)_{\mu\nu} = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{e^{ipx}}{p^2}\Big\{(\delta_{\mu\nu}p^2 - p_\mu p_\nu)\,e^{-\tau p^2} + p_\mu p_\nu\Big\}$$

Lüscher & Weisz, JHEP 1102 (2011) 51
Lüscher, JHEP 04 (2013) 123

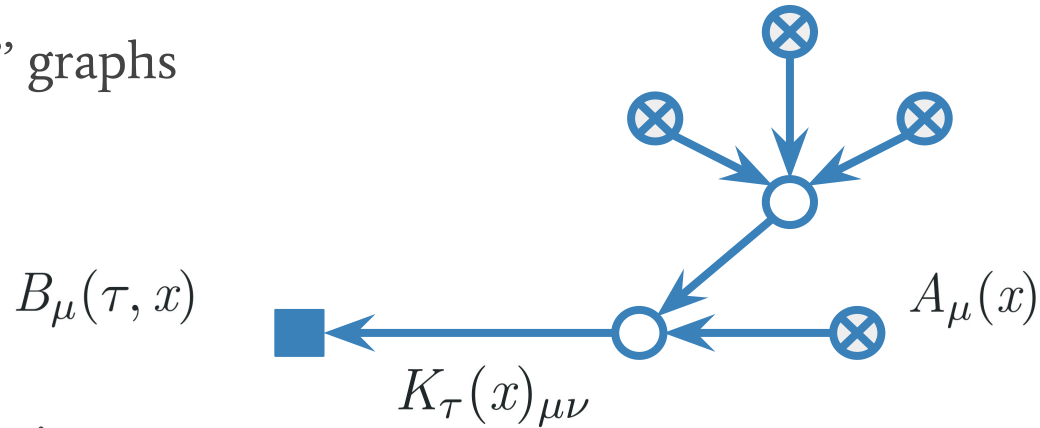
Directed "tree" graphs



Two-point function

$$\left\langle B^a_\mu(\tau, x) B^b_\nu(\sigma, y) \right\rangle =$$

Directed ``tree'' graphs



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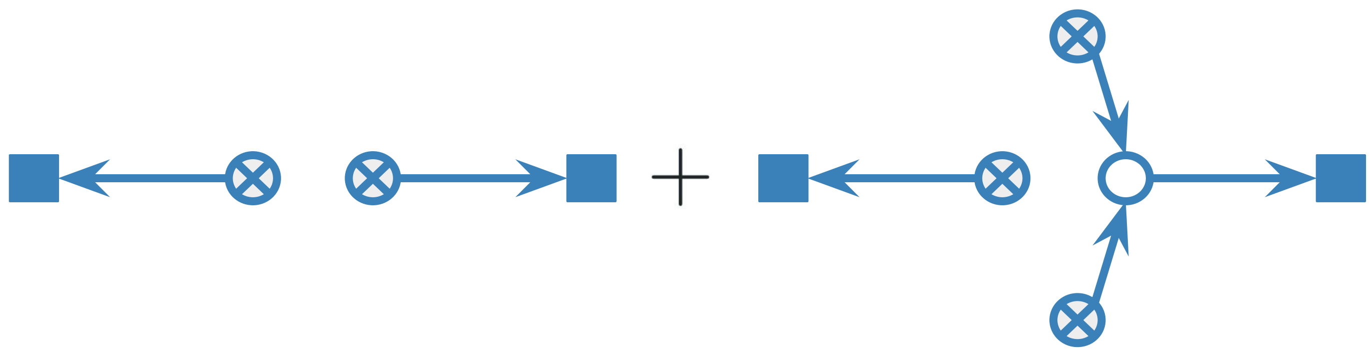
$\left\langle B^a_\mu(\tau,x) B^b_\nu(\sigma,y) \right\rangle =$



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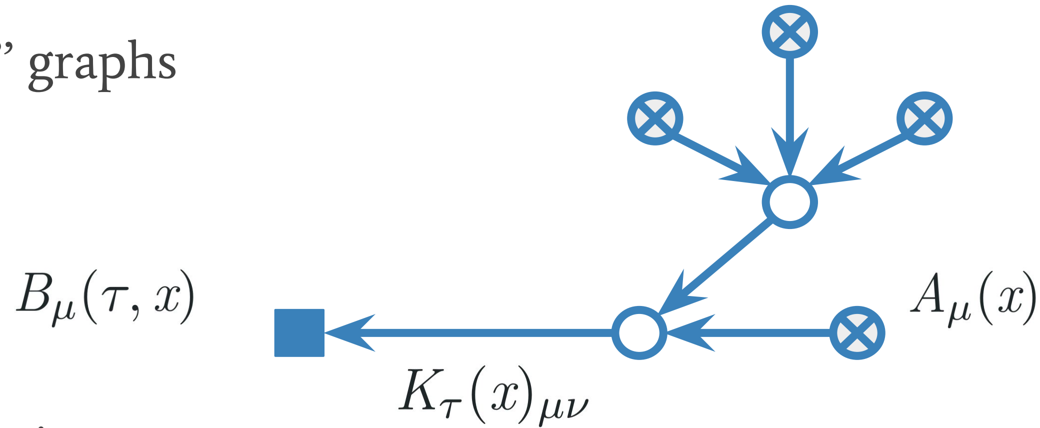

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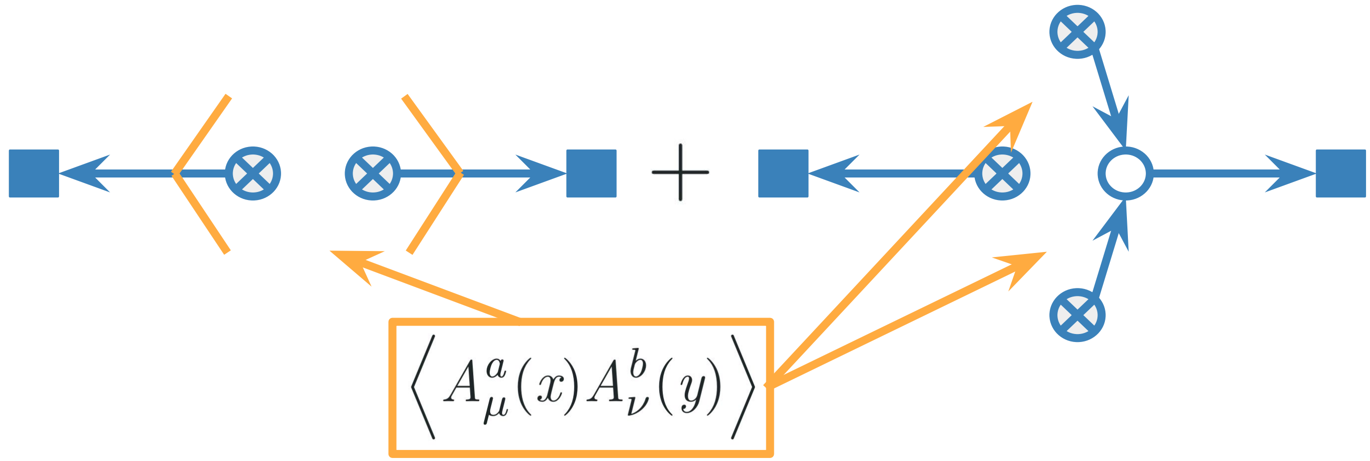

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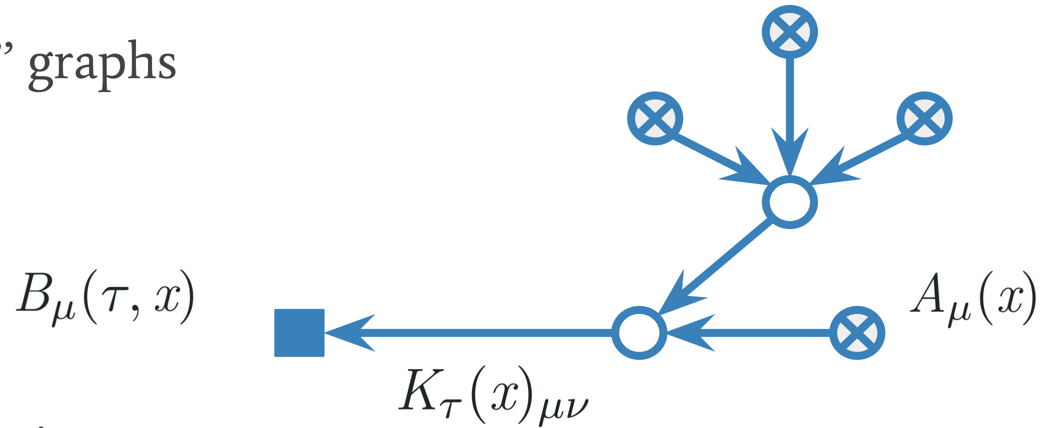

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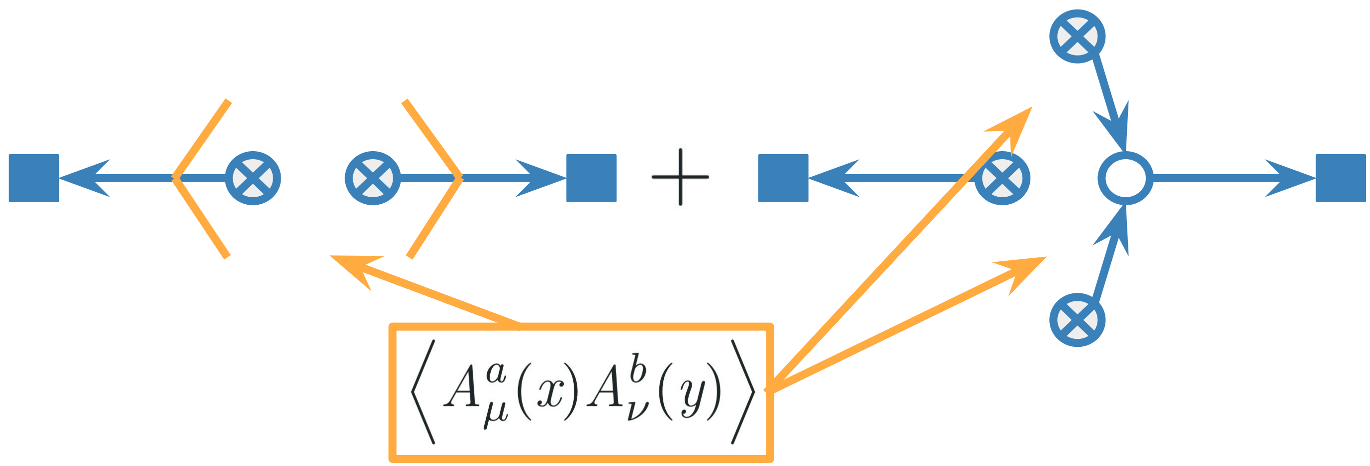
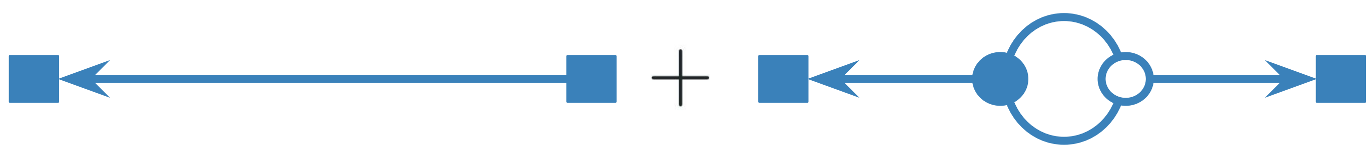


$$\left\langle B^a_\mu(\tau, x) B^b_\nu(\sigma, y) \right\rangle =$$



$$\left\langle \widetilde{B}^a_\mu(t, p) \widetilde{B}^b_\nu(s, q) \right\rangle = (2\pi)^4 \delta(p+q) \delta^{ab} \delta_{\mu\nu} \frac{e^{-(s+t)p^2}}{p^2} + \mathcal{O}(g^2)$$

Smearing removes power-divergent mixing in the continuum, at the expense of introducing a new scale

CJM, PoS(Lattice2015) 052
CJM & K. Orginos, PRD 91 (2015) 074513
CJM & K. Orginos, PoS(Lattice2014) 330

Consider twist-2 operators

$$\mathcal{T}_{\mu_1\ldots\mu_n}(x) = \phi(x)\partial_{\mu_1}\ldots\partial_{\mu_n}\phi(x) - \text{traces}$$

Example: continuum matrix element

$$\langle\,\Omega\,|\,\phi^2(0)\cdot\phi(0)\partial_\mu\partial_\nu\phi(0)\,|\,\Omega\,\rangle = 0$$

On the lattice

$$\langle\Omega|\phi^2(0)\cdot\phi(0)\nabla_\mu\nabla_\nu\phi(0)|\Omega\rangle = -\frac{\delta_{\mu\nu}}{32a^2} + \mathcal{O}(a^0,\lambda)$$

With smeared degrees of freedom

$$\langle\,\Omega\,|\,\overline{\phi}^2(\tau,0)\cdot\overline{\phi}(\tau,0)\nabla_\mu\nabla_\nu\overline{\phi}(\tau,0)\,|\,\Omega\,\rangle = -\frac{\delta_{\mu\nu}}{256\pi^2\tau} + \mathcal{O}(a^0,\lambda)$$

Modify the operator product expansion to account for new scale

Dawson *et al.*, Nucl. Phys. B 514 (1998) 313
Detmold & Lin, Phys. Rev. D 73 (2006) 014501

Wilson's idea: operator product expansion (OPE)

Wilson, Phys. Rev. 179 (1969) 1499

nonlocal operator ~ (perturbative) coefficients x local operators

For example, in free scalar field theory

$$\phi(x)\phi(0) = \frac{1}{4\pi x^2}\mathbb{I} + \phi^2(0) + \mathcal{O}(x)$$

[here the OPE is just a Laurent expansion]

Wilson's idea: operator product expansion (OPE)

Wilson, Phys. Rev. 179 (1969) 1499

nonlocal operator ~ (perturbative) coefficients x local operators

For example, in free scalar field theory

$$\phi(x)\phi(0) = \frac{1}{4\pi x^2}\mathbb{I} + \phi^2(0) + \mathcal{O}(x)$$

Interactions modify Wilson coefficients

$$\phi(x)\phi(0) = \frac{1}{4\pi x^2}\left(1 + a_{\mathbb{I}}\log(\mu^2 x^2)\ldots\right)\mathbb{I} + \left(1 + a_{\phi^2}\log(\mu^2 x^2)\ldots\right)\phi^2(0,\mu) + \mathcal{O}(x)$$

... but not their leading-x behaviour

Operator relation

$$\langle\Omega|\mathcal{O}(x)\widetilde{\phi}(p_1)\ldots\widetilde{\phi}(p_n)|\Omega\rangle \overset{x\to 0}{\sim} \sum_k c_k(x,\mu)\langle\Omega|\mathcal{O}^{(k)}_{\mathrm{R}}(x,\mu)\widetilde{\phi}(p_1)\ldots\widetilde{\phi}(p_n)|\Omega\rangle$$

Replace local operators

nonlocal operator ~ (perturbative) coefficients x local operators

$$\mathcal{O}(x) \overset{x\to 0}{\sim} \sum_k c_k(x,\mu)\mathcal{O}_{\mathrm{R}}^{(k)}(0,\mu) + \ldots$$

with **locally smeared operators**

nonlocal operator ~ (perturbative) coefficients x locally smeared operators

$$\mathcal{O}(x) \overset{x\to 0}{\sim} \sum_k d_k(x,\mu,\tau)\overline{\mathcal{O}}^{(k)}(0,\mu,\tau) + \ldots$$

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$$\mathcal{O}(x) \overset{x\to 0}{\sim} \sum_k c_k(x,\mu)\mathcal{O}_{\mathrm{R}}^{(k)}(0,\mu) + \dots$$

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nonlocal operator ~ (perturbative) coefficients x locally smeared operators

$$\mathcal{O}(x) \overset{x\to 0}{\sim} \sum_k d_k(x,\mu,\tau)\overline{\mathcal{O}}^{(k)}(0,\mu,\tau) + \dots$$

Our example

$$\phi(x)\phi(0) = c_{\mathbb{I}}(x,\mu)\mathbb{I} + c_{\phi^2}(x,\mu)\phi^2(0,\mu) + \mathcal{O}(x)$$

$$\phi(x)\phi(0) = d_{\mathbb{I}}(x,\mu,\tau)\mathbb{I} + d_{\overline{\phi}^2}(x,\mu,\tau)\overline{\phi}^2(0,\mu,\tau) + \mathcal{O}(x,\tau)$$

Calculate Wilson coefficients in standard manner:

$$\phi(x)\phi(0) = d_{\mathbb{I}}(x, \mu, \tau)\mathbb{I} + d_{\overline{\phi}^2}(x, \mu, \tau)\overline{\phi}^2(0, \mu, \tau) + \mathcal{O}(x, \tau)$$

Rearrange sOPE and work at tree-level and expand to order m^2

$$d_{\mathbb{I}}^{(0)}(x, \tau) = \Big\{ \langle \Omega | \phi(x)\phi(0) | \Omega \rangle - \langle \Omega | \phi^2(0, \tau) | \Omega \rangle \Big\}_{\mathcal{O}(\lambda^0, m^2)}$$

[Or graphically]

So

$$d_{\mathbb{I}}^{(0)}(x, \tau) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{ik\cdot x} - e^{-2k^2\tau}}{k^2 + m^2}\bigg|_{\mathcal{O}(\lambda^0, m^2)} = \frac{1}{4\pi x^2}\left\{1 - \frac{x^2}{8\tau} + \frac{m^2 x^2}{4}\left[\gamma_{\mathrm{E}} - 1 + \log\left(\frac{x^2}{8\tau}\right)\right]\right\}$$

Compare to the Wilson coefficient in the original OPE

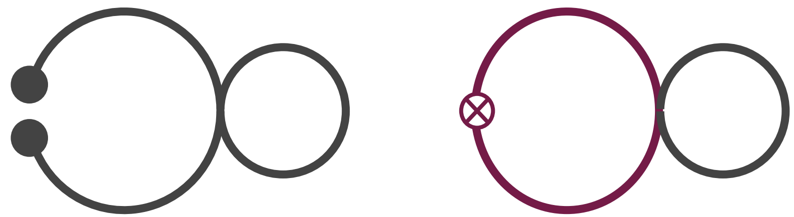
$$c_{\mathbb{I}}^{\overline{MS}}(x, \mu) = \frac{1}{4\pi x^2}\left\{1 + \frac{m^2 x^2}{4}\left[1 + 2\gamma_{\mathrm{E}} + \log\left(\frac{\mu^2 x^2}{16}\right)\right]\right\}$$

Beyond tree-level things get only slightly trickier...

One loop calculation proceeds similarly

$$d_{\mathbb{I}}^{(1)}(x,\tau) = \Big\{ \langle \Omega | \phi(x)\phi(0) | \Omega \rangle - \langle \Omega | \phi^2(0,\tau) | \Omega \rangle \Big\}_{\mathcal{O}(\lambda, m^2)}$$

[Or graphically]



Thus

$$d_{\mathbb{I}}^{(1)}(x,\mu,\tau) = \left\{ \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \frac{e^{ik_1\cdot x} - e^{-k_1^2\tau}}{k_1^2 + m^2} \left[1 - \frac{\lambda}{2} \int \frac{\mathrm{d}^4 k_2}{(2\pi)^4} \frac{1}{k_2^2 + m^2} \right] \right\}_{\mathcal{O}(m^2)}$$

$$= \frac{1}{4\pi x^2} \left\{ 1 - \frac{x^2}{8\tau} + \frac{m_{\mathrm{R}}^2 x^2}{4} \left[\gamma_{\mathrm{E}} - 1 + \log\left(\frac{x^2}{8\tau} \right) \right] \right\}$$

For the leading connected contribution

$$d^{(1)}_{\overline{\phi}^2}(x,\tau) = \Big\{ \langle \Omega | \phi(x)\phi(0)\widetilde{\phi}(p_1)\widetilde{\phi}(p_2)|\Omega\rangle - \langle \Omega|\phi^2(0,\tau)\tilde{\phi}(p_1)\tilde{\phi}(p_2)|\Omega\rangle \Big\}_{\mathcal{O}(\lambda, m^0)}$$

[Or graphically]

$$d^{(1)}_{\overline{\phi}^2}(x,\tau) = \frac{1}{(p_1^2+m^2)(p_2^2+m^2)}\Big\{ 1 - \frac{\lambda}{2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{ik\cdot x} - e^{-(k^2+(k-p_1-p_2)^2)\tau}}{(k^2+m^2)((k-p_1-p_2)^2+m^2)} \Big\}_{\mathcal{O}(m^0)}$$

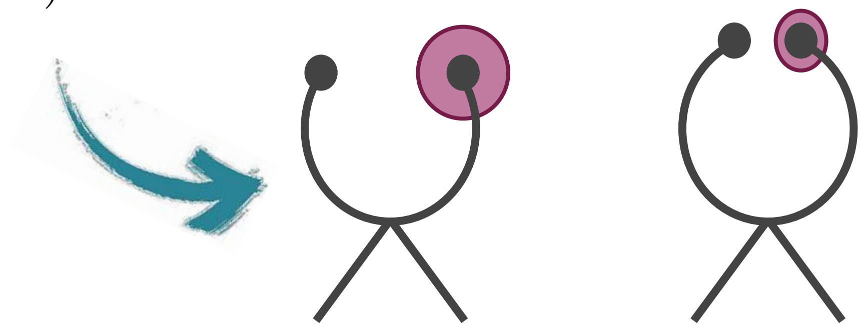
$$d_{\bar{\phi}^2}(x,\tau) = \frac{1}{(p_1^2+m^2)(p_2^2+m^2)}\left\{1-\frac{\lambda}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{e^{ik\cdot x}-e^{-(k^2+(k-p_1-p_2)^2)\tau}}{(k^2+m^2)((k-p_1-p_2)^2+m^2)}\right\}_{\mathcal{O}(m^0)}$$

Derivative:

$$\lambda\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{q\big(e^{ik\cdot x}-e^{-(k^2+q^2)\tau}\big)}{(k^2+m^2)(q^2+m^2)}$$

Convergent: small spacetime limit is well-defined and vanishes if

$$\lim_{x\to 0}\lambda\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{q\big(e^{ik\cdot x}-e^{-(k^2+q^2)\kappa^2x^2}\big)}{(k^2+m^2)(q^2+m^2)}=0$$



$$d_{\bar{\phi}^2}(x,\tau) = \frac{1}{(p_1^2+m^2)(p_2^2+m^2)}\left\{1 - \frac{\lambda}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{e^{ik\cdot x} - e^{-(k^2+(k-p_1-p_2)^2)\tau}}{(k^2+m^2)((k-p_1-p_2)^2+m^2)}\right\}_{\mathcal{O}(m^0)}$$

Derivative:

$$\lambda \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{q\left(e^{ik\cdot x} - e^{-(k^2+q^2)\tau}\right)}{(k^2+m^2)(q^2+m^2)}$$

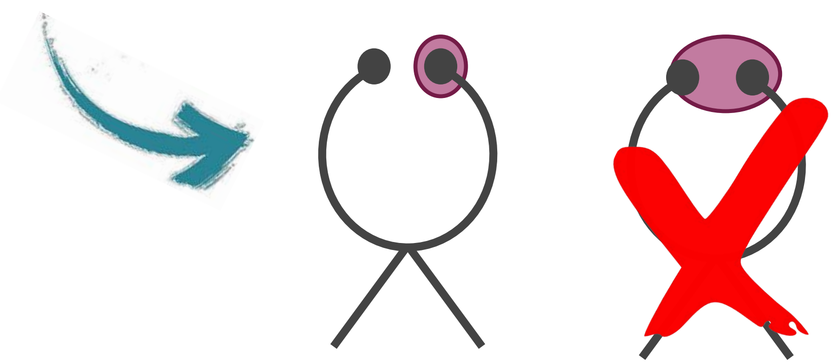
Convergent: small spacetime limit is well-defined and vanishes if

$$\lim_{x\to 0} \lambda \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{q\left(e^{ik\cdot x} - e^{-(k^2+q^2)\kappa^2 x^2}\right)}{(k^2+m^2)(q^2+m^2)} = 0$$

Lüscher & Weisz, JHEP 1102 (2011) 51
Suzuki, PTEP (2013) 083B03
Makino & Suzuki, PTEP (2014) 063B02

or apply the small flow-time expansion

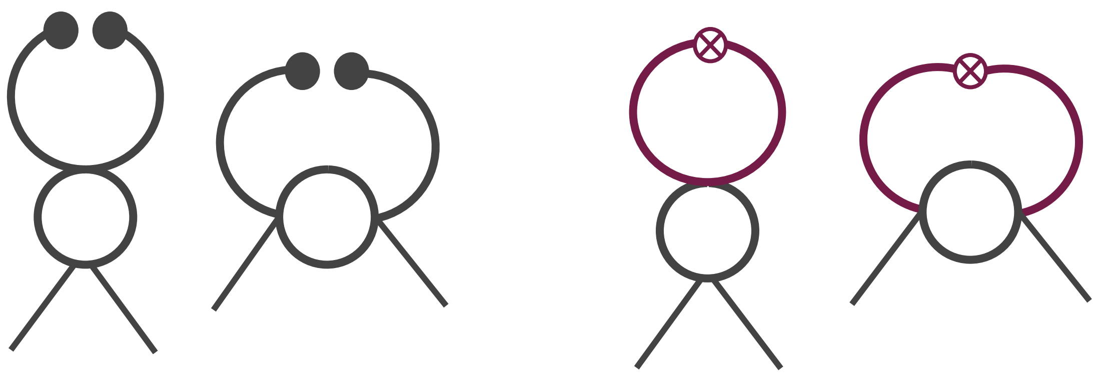
$$\lim_{x\to 0} \lambda \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{q\left(e^{ik\cdot x} - 1\right)}{(k^2+m^2)(q^2+m^2)} = 0 + \mathcal{O}(\tau)$$



At one loop

$$d_{\bar{\phi}^2}^{(1)}(x,\tau) = 1 + \frac{\lambda}{32\pi^2}\left[\gamma_{\mathrm{E}} - 1 + \log\left(\frac{x^2}{8\tau}\right)\right]$$

At two loops



Leading to

$$d_{\bar{\phi}^2}^{(2)}(x,\mu,\tau) = 1 + \frac{\lambda_{\mathrm{R}}}{32\pi^2}\left[\gamma_{\mathrm{E}} - 1 + \log\left(\frac{x^2}{8\tau}\right)\right]$$

Renormalisation group equations for connected Green functions

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} = \mu \frac{\partial}{\partial \mu}\bigg|_{\lambda_{\mathrm{R}}, m_{\mathrm{R}}} + \beta \frac{\partial}{\partial \lambda_{\mathrm{R}}}\bigg|_{\mu, m_{\mathrm{R}}} - \gamma_m m_{\mathrm{R}} \frac{\partial}{\partial m_{\mathrm{R}}}\bigg|_{\mu, \lambda_{\mathrm{R}}}$$

Then

$$\left[\mu \frac{\mathrm{d}}{\mathrm{d}\mu} + N\gamma\right] \langle \Omega | \widetilde{\phi}_{\mathrm{R}}(p_1) \ldots \widetilde{\phi}_{\mathrm{R}}(p_N) | \Omega \rangle = 0$$

$$\left[\mu \frac{\mathrm{d}}{\mathrm{d}\mu} + N\gamma - \gamma_m\right] \langle \Omega | \left[\phi^2(0, \mu)\right]_{\mathrm{R}} \widetilde{\phi}_{\mathrm{R}}(p_1) \ldots \widetilde{\phi}_{\mathrm{R}}(p_N) | \Omega \rangle = 0$$

Applying

$$\left[\mu \frac{\mathrm{d}}{\mathrm{d}\mu} + (N+2)\,\gamma\right]$$

to our example OPE

$$\langle \Omega | \widetilde{\phi}_{\mathrm{R}}(p_1) \ldots \widetilde{\phi}_{\mathrm{R}}(p_{N+2}) | \Omega \rangle = c_{\phi^2}(x, \mu) \langle \Omega | \left[\phi^2(0, \mu)\right]_{\mathrm{R}} \widetilde{\phi}_{\mathrm{R}}(p_1) \ldots \widetilde{\phi}_{\mathrm{R}}(p_N) | \Omega \rangle + \mathcal{O}(x)$$

we obtain

$$\left[\mu \frac{\mathrm{d}}{\mathrm{d}\mu} + 2\,(\gamma + \gamma_m)\right] c_{\phi^2}(x, \mu) = 0$$

For the sOPE we now have two scales

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu} \to \mu\frac{\mathrm{d}}{\mathrm{d}\mu} - \tau\frac{\mathrm{d}}{\mathrm{d}\tau}$$

Use the small flow time expansion

$$\left[\phi^2(0,\mu)\right]_{\mathrm{R}} = \mathcal{Z}_{\overline{\phi}^2}(\mu,\tau)\overline{\phi}^2(0,\tau) + \mathcal{O}(\tau)$$

where

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\log\left[\mathcal{Z}_{\overline{\phi}^2}(\mu,\tau)\right] = 2\gamma_m$$

and we define

$$\zeta_{\overline{\phi}^2} = \frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\log\left[\mathcal{Z}_{\overline{\phi}^2}(\mu,\tau)\right]$$

Eventually we obtain

$$\left[\mu\frac{\mathrm{d}}{\mathrm{d}\mu} - \tau\frac{\mathrm{d}}{\mathrm{d}\tau} + 2\left(\gamma - \zeta_{\overline{\phi}^2}\right)\right] d_{\overline{\phi}^2}(x,\mu,\tau) = 0$$

Analogous equations apply to the matrix elements

Summary

1. Power-divergent mixing restricts lattice calculations to low moments
2. Smearing removes power-divergent mixing in the continuum, at the expense of introducing a new scale
3. Modify the operator product expansion to account for new scale

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2. Smearing removes power-divergent mixing in the continuum, at the expense of introducing a new scale
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Looking forward

1. How (im)practical is this, really?
2. Gradient flow sum rules [with Herbert Neuberger]?
3. Other ways to deal with the extra scale?

Thank you

chris.monahan@rutgers.edu

IAS TUM
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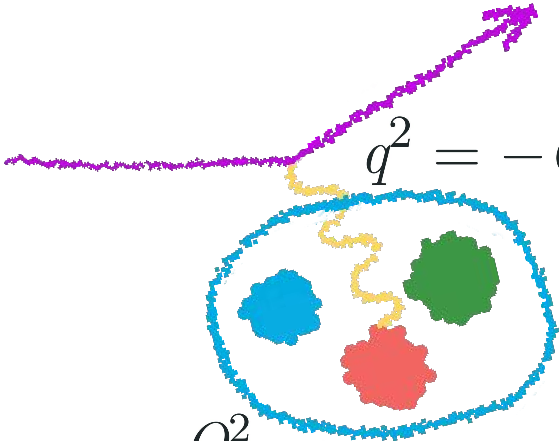
Deep inelastic scattering



$$q^2 = -Q^2$$

$$x = \frac{Q^2}{2P \cdot q}$$

Decompose cross-section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega \mathrm{d}E'} = \frac{e^4}{16\pi^2 Q^4} \ell^{\mu\nu} W_{\mu\nu}$$

Hadronic tensor

$$W_{\mu\nu}(p,q) = \frac{1}{4\pi} \int \mathrm{d}^4x \, e^{iq\cdot x} \langle p, \lambda' \,|\, [j_\mu(x), j_\nu(x)] \,|\, p, \lambda \rangle$$

Express in terms of structure functions F_1, F_2, g_1, g_2

$$F(x, Q^2) = \int \mathrm{d}y \, C\left(\frac{x}{y}, \frac{Q^2}{\mu^2}\right) f_{q/N}(x, \mu^2)$$

(Light front) parton distributions universal

$$f_{q/N}(x, Q^2) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \mathrm{d}y^- \, e^{-iy^- p^+} \langle N \,|\, \overline{\psi}(0^+, y^-, 0_\mathrm{T}) \gamma_+ U(y^-, 0) \, \psi(0) \,|\, N \rangle$$

Relate hadronic tensor to forward Compton amplitude

$$W_{\mu\nu} = \frac{1}{2\pi}\mathrm{Im}\{T_{\mu\nu}\}$$

Operator product expansion generates "twist" (dimension - spin) expansion

Twist-2 operators dominate in Bjorken limit

$$\overline{\psi}\gamma_{\{\mu_1}\overleftrightarrow{D}_{\mu_2}\ldots\overleftrightarrow{D}_{\mu_n\}}\psi - \text{traces}$$

Mellin moments

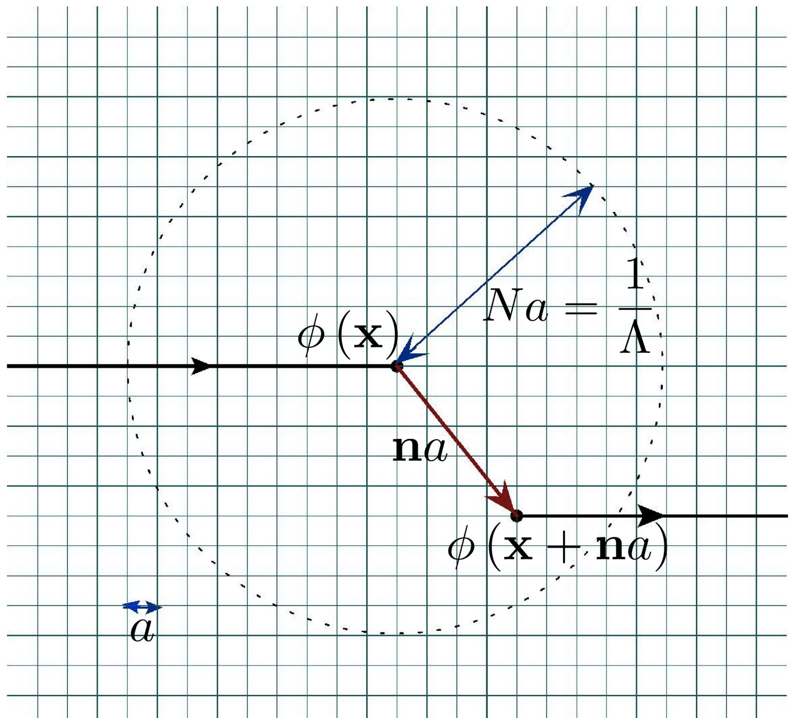
$$\langle x^n\rangle_{f_{q/N}} = \int_{-1}^{1}\mathrm{d}x\, x^n f_{q/N}(x)$$

$$2\langle x^n\rangle_{f_{q/N}}P_{\mu_1}\cdots P_{\mu_n} = \frac{1}{2}\langle N(P)|\overline{\psi}\gamma_{\{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n\}}\psi|N(P)\rangle$$

Wick rotation of moments is trivial

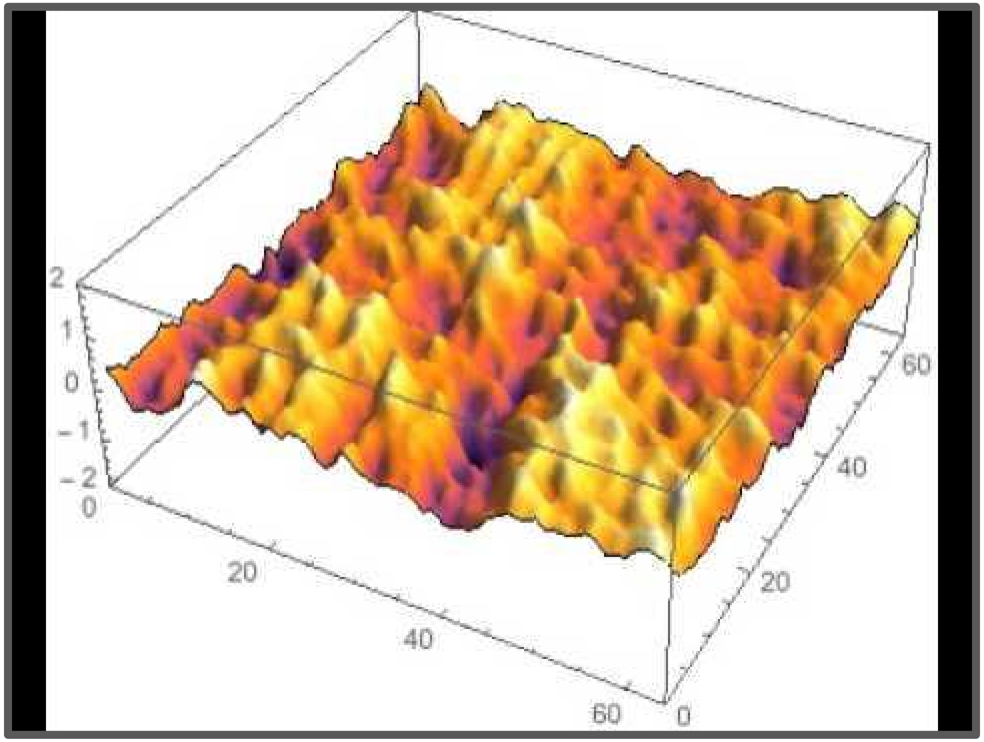
... however...

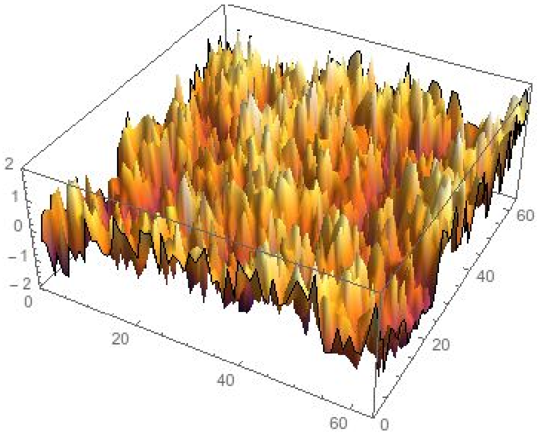
“Smearing” partially restores rotational symmetry: **suppresses operator mixing**



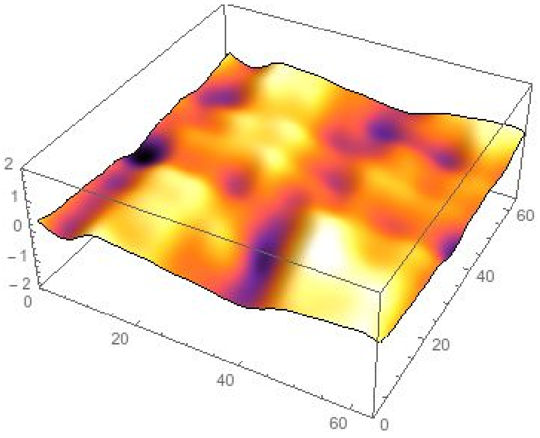
Davoudi & Savage, Phys. Rev. D 86 (2012) 054505

Construct operators with improved continuum limits





Flow-time



CJM, PoS(Lattice2015) 052

Renormalisation group equation

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu} \to \mu\frac{\mathrm{d}}{\mathrm{d}\mu} - 2\tau\frac{\mathrm{d}}{\mathrm{d}\tau}$$

For sufficiently small flow times

$$[\phi^2(0)]_{\mathrm{R}} = \mathcal{Z}_{\phi^2}(\tau,\mu)\phi^2(\tau,0) \qquad \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\log\left[\mathcal{Z}_{\phi^2}(\tau,\mu^2)\right] = 2\gamma_m$$

Perturbative coefficient obeys

$$\left[\mu\frac{\mathrm{d}}{\mathrm{d}\mu} - 2\tau\frac{\mathrm{d}}{\mathrm{d}\tau} + 2(\zeta_{\phi^2} - \gamma)\right] d_{\phi^2} = 0 \qquad \zeta_{\phi^2} = \tau\frac{\mathrm{d}}{\mathrm{d}\tau}\log\left[\mathcal{Z}_{\phi^2}(\tau,\mu^2)\right]$$

Corresponding nonperturbative matrix elements satisfy

$$\left[\mu\frac{\mathrm{d}}{\mathrm{d}\mu} - 2\tau\frac{\mathrm{d}}{\mathrm{d}\tau} + 2(\zeta_{\phi^2} + \gamma)\right]\langle\Omega|\phi^2(\tau,0)\tilde{\phi}(p_1)\tilde{\phi}(p_2)|\Omega\rangle = 0$$

Following a line of constant physics

$$\left[\mu\frac{\mathrm{d}}{\mathrm{d}\mu} + \zeta_{\phi^2} + \gamma\right]\langle\Omega|\phi^2(1/\mu^2,0)\tilde{\phi}(p_1)\tilde{\phi}(p_2)|\Omega\rangle = 0$$

Makino & Suzuki, PTEP (2015) 033B08
Makino *et al.*, PTEP (2015) 043B07
Aoki *et al.*, JHEP 1504 (2015) 156
Kikuchi & Onogi, JHEP 1411 (2014) 094

Gradient flow in 2D O(3) model

$$\frac{\partial n^i(\tau,x)}{\partial \tau} = \left[\delta^{ij} - n^i(\tau,x)n^j(\tau,x)\right]\partial^2 n^j(\tau,x) \qquad n^i(\tau=0,x) = n^i(x)$$

$$n^i(\tau,x) = \pi^i(\tau,x) \quad \text{for} \quad i=1,2 \qquad n^3(\tau,x) = \sqrt{1-\pi^i(\tau,x)\pi^i(\tau,x)}$$

Exact solution no longer possible: generate iterative tree-level expansion

$$n^i(\tau,x) = \int \mathrm{d}^2y \int \frac{\mathrm{d}^2p}{(2\pi)^2}\, e^{ip\cdot(x-y)} \left[e^{-\tau p^2} n^i(y) - \int_0^\tau \mathrm{d}s\, e^{-sp^2} R^i(s,y)\right]$$

$$R^i(s,y) = n^i(s,y)n^j(s,x)\partial^2 n^j(s,x)$$

Interactions occur in the bulk, *i.e.* at non-zero flow time, but no closed loops

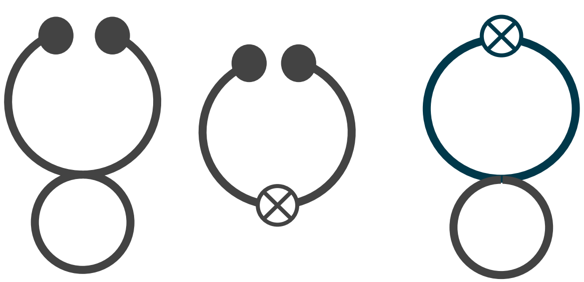
Consider

$$\pi(x)\pi(0) = \frac{b_{\mathbb{I}}}{4\pi}\mathbb{I} + b_{\pi^2}\pi^2(\tau, 0) + b_{\partial_\mu} x^\mu \partial_\mu \pi^2(\tau, 0) + \ldots$$

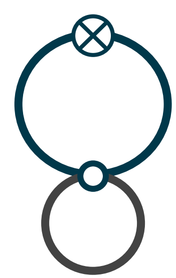
One loop calculations (almost) as straightforward as 4D φ^4 scalar field theory

Two loops - interactions complicate the picture

I. quantum interactions



2. tree interactions



but still no flow loops

Lattice determinations: nucleon structure

Meson distribution amplitudes

- quenched — Martinelli & Sachrajda, PLB 1 (1987) 184; Martinelli & Sachrajda, NPB 306 (1988) 805
- unquenched — Best et al, PRD 56 (1997) 2743

Nucleon

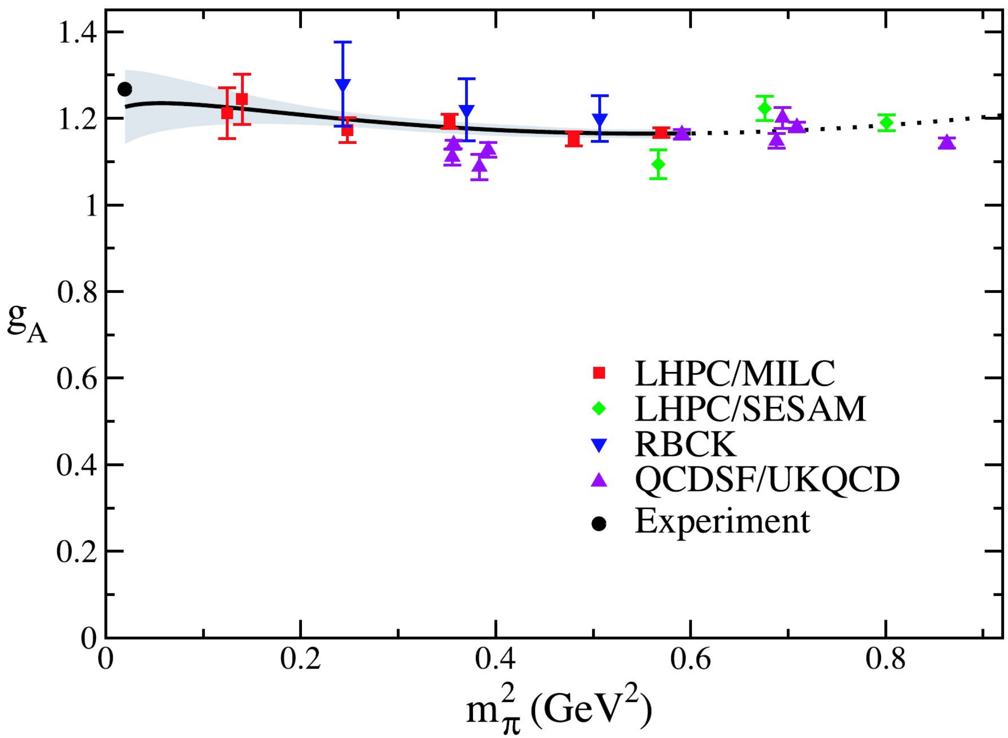
- axial charge — Edwards et al, PRL 96 (2006) 052001; Capitani et al, PRD 86 (2012) 074502; Horsley *et al*, PLB 732 (2014) 41
- unpolarised — Gockeler et al, PRD 53 (1996) 2317
- polarised — Gockeler *et al*, PRD 53 (1996) 2317
- higher twist contributions — Capitani *et al*, NPB (Proc. Suppl.) 79 (1999) 179
- transverse momentum distributions — Y. Zhao, arXiv/1506.08832; Musch *et al*, PRD 83 (2011) 094507
- generalised parton distributions — Hagler et al, PRL 93 (2004) 112001; Gockeler et al, PRL 92 (2004) 042002

W. Bietenholz et al, PoS LATTTICE(2009) 138

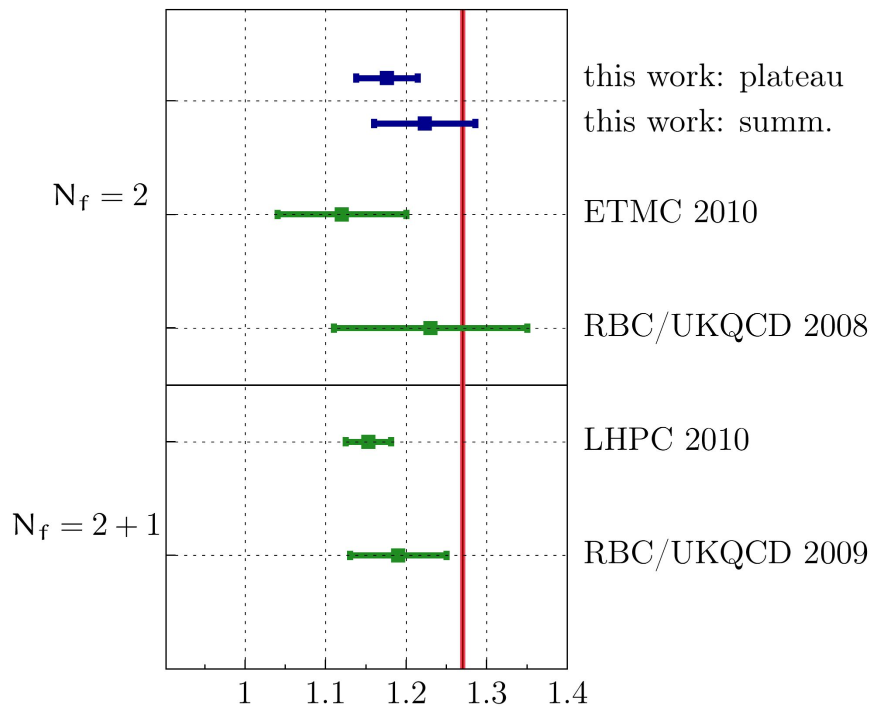
Nucleon axial charge

$$\langle x^0 \rangle_{\Delta q} = \int_0^1 \mathrm{d}x \, [\Delta q(x) + \Delta \bar{q}(x)]$$

$$\Delta q(x) = q_\uparrow(x) - q_\downarrow(x)$$



Edwards *et al*, Phys. Rev. Lett. 96 (2006) 052001



Capitani *et al*, Phys. Rev. D 86 (2012) 074502

Direct determination of PDFs: LaMET

Relate PDFs

$$q(x,\mu^2) = \int \frac{\mathrm{d}\xi^-}{4\pi} e^{-ix\xi^- P^+} \langle P|\overline{\psi}(\xi^-)\gamma^+ e^{-ig\int_0^{\xi^-} \mathrm{d}\eta^- A^+(\eta^-)}\psi(0)|P\rangle$$

X. Ji et al, PRD 91 (2015) 074009
X. Ji, Sc. China (2014)
X. Ji, PRL 110 (2013) 262002

to "quasi"-distributions

$$\overline{q}(x,\mu^2,P^z) = \int \frac{\mathrm{d}z}{4\pi} e^{izk^z} \langle P|\overline{\psi}(z)\gamma^z e^{-ig\int_0^z \mathrm{d}z' A^z(z')}\psi(0)|P\rangle + \mathcal{O}\left(\Lambda^2_{\mathrm{QCD}}/(P^z)^2, M^2/(P^z)^2\right)$$

via a factorisation formula

$$\overline{q}(x,\mu^2,P^z) = \int_x^1 \frac{\mathrm{d}y}{y} Z\left(\frac{x}{y},\frac{\mu}{P^z}\right) q(y,\mu^2) + \mathcal{O}\left(\Lambda^2_{\mathrm{QCD}}/(P^z)^2, M^2/(P^z)^2\right)$$

X. Ji & J.-H. Zhang, PRD 92 (2015) 034006
X. Ji et al, arXiv/1506.00248
X. Xiong et al, PRD 90 (2014) 014051

Requires renormalisation of nonlocal operators

Some progress towards this via HQET at NLO

- relation to OPE-based approaches?

W. Detmold & C.J.D. Lin, PRD 73 (2006) 014501

Initial lattice studies at a single lattice spacing

C. Alexandrou et al, PRD 92 (2015) 014502
H.-W. Lin et al, PRD 91 (2014) 054510

Moments of quark density

$$\langle x^n \rangle_q = \int_0^1 \mathrm{d}x\, x^n (q(x) + (-1)^{n+1} \bar{q}(x)) \qquad q = q_\uparrow + q_\downarrow$$

helicity

$$\langle x^n \rangle_{\Delta q} = \int_0^1 \mathrm{d}x\, x^n (\Delta q(x) + (-1)^{n} \Delta\bar{q}(x)) \qquad \Delta q = q_\uparrow - q_\downarrow$$

and transversity

$$\langle x^n \rangle_{\delta q} = \int_0^1 \mathrm{d}x\, x^n (\delta q(x) + (-1)^{n+1} \delta\bar{q}(x)) \qquad \delta q = q_\top - q_\perp$$

Odd moments related to spin-independent structure functions

$$\int_0^1 \mathrm{d}x\, x^{n-1} F_1(x, Q^2) = \frac{1}{2} c_n^{(q)}(Q^2/\mu^2) \sum_f e_f^2 \langle x^{n-1} \rangle_{q_f}(\mu)$$

$$\int_0^1 \mathrm{d}x\, x^{n-2} F_2(x, Q^2) = c_n^{(q)}(Q^2/\mu^2) \sum_f e_f^2 \langle x^{n-1} \rangle_{q_f}(\mu)$$

Even moments related to spin-dependent structure function

$$\int_0^1 \mathrm{d}x\, x^{n} g_1(x, Q^2) = \frac{1}{2} c_n^{(\Delta q)}(Q^2/\mu^2) \sum_f e_f^2 \langle x^{n} \rangle_{\Delta q_f}(\mu)$$

Moments are related to matrix elements of local operators

$$\mathcal{O}^{(q_f)}_{\{\mu_1 \ldots \mu_n\}} = \left(\frac{i}{2}\right)^{n-1} \overline{\psi}^f \gamma_{\{\mu_1} \overset{\leftrightarrow}{D}_{\mu_2} \cdots \overset{\leftrightarrow}{D}_{\mu_n\}} \psi^f$$

$$\mathcal{O}^{(\Delta q_f)}_{\{\sigma\mu_1 \ldots \mu_n\}} = \left(\frac{i}{2}\right)^{n} \overline{\psi}^f \gamma_5 \gamma_{\{\sigma} \overset{\leftrightarrow}{D}_{\mu_1} \cdots \overset{\leftrightarrow}{D}_{\mu_n\}} \psi^f$$

$$\mathcal{O}^{(\delta q_f)}_{\mu\{\nu\mu_1 \ldots \mu_n\}} = \left(\frac{i}{2}\right)^{n} \overline{\psi}^f \gamma_5 \sigma_{\mu\{\nu} \overset{\leftrightarrow}{D}_{\mu_1} \cdots \overset{\leftrightarrow}{D}_{\mu_n\}} \psi^f$$

Via

$$2\langle x^{n-1}\rangle_{q_f} P_{\mu_1} \cdots P_{\mu_n} = \frac{1}{2}\sum_S \langle P, S|\mathcal{O}^{(q_f)}_{\{\mu_1 \ldots \mu_n\}}|P, S\rangle$$

$$\frac{2}{n+1}\langle x^{n}\rangle_{\Delta q_f} S_{\{\sigma} P_{\mu_1} \cdots P_{\mu_n\}} = -\langle P, S|\mathcal{O}^{(\Delta q_f)}_{\{\sigma\mu_1 \ldots \mu_n\}}|P, S\rangle$$

$$\frac{2}{m_N}\langle x^{n}\rangle_{\delta q_f} S_{[\mu} P_{\{\nu]} P_{\mu_1} \cdots P_{\mu_n\}} = \langle P, S|\mathcal{O}^{(\delta q_f)}_{\mu\{\nu\mu_1 \ldots \mu_n\}}|P, S\rangle$$

For Euclidean lattice operators

$$\mathcal{O}^{(q_f)}_{\{\mu_1 \ldots \mu_n\}} = \overline{\psi}^f \gamma_{\{\mu_1} \overleftrightarrow{D}_{\mu_2} \cdots \overleftrightarrow{D}_{\mu_n\}} \psi^f$$

$$\mathcal{O}^{(5)}_{\{\sigma\mu_1 \ldots \mu_n\}} = \overline{\psi}^f \gamma_{\{\sigma} \gamma_5 \overleftrightarrow{D}_{\mu_1} \cdots \overleftrightarrow{D}_{\mu_n\}} \psi^f$$

Lie in same O(4) irrep, but inequivalent reps of H(4)

$$\mathcal{O}^{(q_f)}_{\{14\}} \qquad \mathcal{O}^{(q_f)}_{\{44\}} - \frac{1}{3} \sum_{i=1}^{3} \mathcal{O}^{(q_f)}_{\{ii\}}$$

Lie in same H(4) irrep

$$\mathcal{O}^{(5)}_{\{14\}} \qquad \mathcal{O}^{(5)}_{\{24\}}$$

See, for example, Gockeler et al, PRD 54 (1996) 5705

Second moment operator

$$\mathcal{O}^{(q_f)}_{\{114\}} - \frac{1}{2} \left(\mathcal{O}^{(q_f)}_{\{224\}} + \mathcal{O}^{(q_f)}_{\{334\}} \right)$$

Third moment operator

$$\mathcal{O}^{(q_f)}_{\{1144\}} + \mathcal{O}^{(q_f)}_{\{2233\}} - \mathcal{O}^{(q_f)}_{\{1133\}} - \mathcal{O}^{(q_f)}_{\{2244\}}$$

which mixes with

$$\overline{\psi}^f \sigma_{[\mu\{\nu} \gamma_5 \overleftrightarrow{D}_{\mu_1]} \overleftrightarrow{D}_{\mu_2\}} \psi^f$$