Lattice QCD Study of Excited Hadron Resonances

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Effective Field Theories and Lattice Gauge Theory
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spectroscopy resurgence due to discovery of unexpected charmonium \textit{XYZ} states

GlueX and JLab Hall D search for hybrids, other exotics

\begin{itemize}
  \item \textbf{Spectrum of QCD}
  \item \textbf{C. Morningstar}
  \item \textbf{Excited States}
\end{itemize}
Key Points

- crucial role of interpolating operators for excited-state studies in lattice QCD
- lower-lying multi-hadron levels must be dealt with
- need to handle many quark lines: stochastic LapH method

- level identification using interpolating operators
- finite-volume energies \( \Rightarrow \) hadron resonance properties: masses, decay widths

- focus: large \( 32^3 \) anisotropic lattices, \( m_\pi \sim 240 \text{ MeV} \)
- scattering phase shifts from finite-volume energies
- need for effective Hamiltonian approach
- tetraquark operators
Excited states from correlation matrices

- in finite volume, energies are discrete (neglect wrap-around)
  \[ C_{ij}(t) = \sum_n Z_i^{(n)} Z_j^{(n)*} e^{-E_n t}, \quad Z_j^{(n)} = \langle 0 | O_j | n \rangle \]

- not practical to do fits using above form
- define new correlation matrix \( \widetilde{C}(t) \) using a single rotation
  \[ \widetilde{C}(t) = U^\dagger C(\tau_0)^{-1/2} C(t) C(\tau_0)^{-1/2} U \]
  
  - columns of \( U \) are eigenvectors of \( C(\tau_0)^{-1/2} C(\tau_D) C(\tau_0)^{-1/2} \)
  
  - choose \( \tau_0 \) and \( \tau_D \) large enough so \( \widetilde{C}(t) \) diagonal for \( t > \tau_D \)
  
  - effective energies
    \[ \tilde{m}_{\alpha}^{\text{eff}}(t) = \frac{1}{\Delta t} \ln \left( \frac{\widetilde{C}_{\alpha\alpha}(t)}{\widetilde{C}_{\alpha\alpha}(t + \Delta t)} \right) \]
    
    tend to \( N \) lowest-lying stationary state energies in a channel

- 2-exponential fits to \( \widetilde{C}_{\alpha\alpha}(t) \) yield energies \( E_\alpha \) and overlaps \( Z_j^{(n)} \)
Correlator matrix toy model

- **Theorem:** For every $t \geq 0$, let $\lambda_n(t)$ be the eigenvalues of an $N \times N$ Hermitian correlation matrix $C(t)$ ordered such that $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{N-1}$, then

$$\lim_{t \to \infty} \lambda_n(t) = b_n e^{-E_n t} \left[ 1 + O(e^{-t \Delta_n}) \right],$$

with $b_n > 0$, $\Delta_n = \min_{m \neq n} |E_n - E_m|$.

- **Example:** $N_e = 200$ eigenstates with energies

$$E_0 = 0.20, \quad E_n = E_{n-1} + \frac{0.08}{\sqrt{n}}, \quad n = 1, 2, \ldots, N_e - 1.$$ 

for $N \times N$ correlator matrix, $N = 12$, overlaps

$$Z_j^{(n)} = \frac{(-1)^{j+n}}{1 + 0.05(j - n)^2}.$$
Correlator matrix toy model (con’t)

- toy model $N_e = 200$ with $12 \times 12$ correlator matrix $C(t)$

- left: effective energies of diagonal elements of correlator matrix

- middle: effective energies of eigenvalues of $C(t)$

- right: effective energies of eigenvalues of $C(\tau_0)^{-1/2} C(t) C(\tau_0)^{-1/2}$ for $\tau_0 = 1$
Building blocks for single-hadron operators

- Building blocks: covariantly-displaced LapH-smeared quark fields
- Stout links $\tilde{U}_j(x)$
- Laplacian-Heaviside (LapH) smeared quark fields
  \[
  \tilde{\psi}_{a\alpha}(x) = S_{ab}(x,y) \psi_{b\alpha}(y), \quad S = \Theta \left( \sigma_s^2 + \tilde{\Delta} \right)
  \]
- 3d gauge-covariant Laplacian $\tilde{\Delta}$ in terms of $\tilde{U}$
- Displaced quark fields:
  \[
  q_{Aa\alpha j} = D(j) \tilde{\psi}_{a\alpha}^{(A)}, \quad \bar{q}_{Aa\alpha j} = \bar{\tilde{\psi}}_{a\alpha}^{(A)} \gamma_4 D(j)^\dagger
  \]
- Displacement $D(j)$ is product of smeared links:
  \[
  D(j)(x,x') = \tilde{U}_{j_1}(x) \tilde{U}_{j_2}(x+d_2) \tilde{U}_{j_3}(x+d_3) \cdots \tilde{U}_{j_p}(x+d_p) \delta_{x', x+d_{p+1}}
  \]
- To good approximation, LapH smearing operator is
  \[
  S = V_s V_s^\dagger
  \]
- Columns of matrix $V_s$ are eigenvectors of $\tilde{\Delta}$
Extended operators for single hadrons

- quark displacements build up orbital, radial structure

Meson configurations

\[ \Phi_{AB}(p, t) = \sum_x e^{ip \cdot (x + \frac{1}{2} (d_{\alpha} + d_{\beta}))} \delta_{ab} \bar{q}_B^B(x, t) q_A^A(x, t) \]

\[ \Phi_{ABC}(p, t) = \sum_x e^{ip \cdot x} \varepsilon_{abc} \bar{q}_C^C(x, t) \bar{q}_B^B(x, t) \bar{q}_A^A(x, t) \]

- group-theory projections onto irreps of lattice symmetry group

\[ \overline{M}_l(t) = c_{\alpha\beta}^{(l)*} \Phi_{\alpha\beta}(t) \quad \quad \overline{B}_l(t) = c_{\alpha\beta\gamma}^{(l)*} \Phi_{\alpha\beta\gamma}(t) \]

- definite momentum \( p \), irreps of little group of \( p \)
Importance of smeared fields

- effective masses of 3 selected nucleon operators shown
- noise reduction of displaced-operators from link smearing $n_{\rho\rho} = 2.5, n_\rho = 16$
- quark-field smearing $\sigma_s = 4.0, n_\sigma = 32$
  reduces excited-state contamination
Two-hadron operators

- Our approach: superposition of products of single-hadron operators of definite momenta

\[
C^{I_3aI_3b}_{p_a\lambda_a; p_b\lambda_b} B^{I_3aS_a}_{p_a\Lambda_a\lambda_a i_a} B^{I_3bS_b}_{p_b\Lambda_b\lambda_b i_b}
\]

- Fixed total momentum \( p = p_a + p_b \), fixed \( \Lambda_a, i_a, \Lambda_b, i_b \)

- Group-theory projections onto little group of \( p \) and isospin irreps

- Crucial to know and fix all phases of single-hadron operators for all momenta
  - Each class, choose reference direction \( p_{\text{ref}} \)
  - Each \( p \), select one reference rotation \( R^p_{\text{ref}} \) that transforms \( p_{\text{ref}} \) into \( p \)

- Efficient creating large numbers of two-hadron operators

- Generalizes to three, four, ... hadron operators
temporal correlations involving our two-hadron operators need
- slice-to-slice quark lines (from all spatial sites on a time slice to all spatial sites on another time slice)
- sink-to-sink quark lines

isoscalar mesons also require sink-to-sink quark lines

solution: the stochastic LapH method!
Stochastic estimation of quark propagators

- do not need exact inverse of Dirac matrix $K[U]$
- introduce $Z_4$ noise vectors $\eta$ in the LapH subspace

$$\eta_{\alpha k}(t), \quad t = \text{time}, \ \alpha = \text{spin}, \ k = \text{eigenvector number}$$

- solve $K[U]X^{(r)} = \eta^{(r)}$ for each of $N_R$ noise vectors $\eta^{(r)}$, then obtain a Monte Carlo estimate of all elements of $K^{-1}$

$$K^{-1}_{ij} \approx \frac{1}{N_R} \sum_{r=1}^{N_R} X_i^{(r)} \eta_j^{(r)*}$$

- variance reduction using noise dilution
- dilution introduces projectors $P^{(a)}$, then define

$$\eta^{[a]} = P^{(a)} \eta, \quad X^{[a]} = K^{-1} \eta^{[a]}$$

to obtain Monte Carlo estimate with drastically reduced variance

$$K^{-1}_{ij} \approx \frac{1}{N_R} \sum_{r=1}^{N_R} \sum_a X_i^{(r)[a]} \eta_j^{(r)[a]*}$$
The effectiveness of stochastic LapH

- comparing use of lattice noise vs noise in LapH subspace
- $N_D$ is number of solutions to $Kx = y$
Correlators and quark line diagrams

- **baryon correlator**

\[
C_{\bar{l}l} \approx \frac{1}{N_R} \sum_r \sum_{dAdbdc} B_l^{(r)[dAdbdc]} (\varphi^A, \varphi^B, \varphi^C) B_{\bar{l}}^{(r)[dAdbdc]} (\bar{\varrho}^A, \bar{\varrho}^B, \bar{\varrho}^C)^* 
\]

- express diagrammatically

- **meson correlator**

\[
- \begin{array}{c}
\begin{array}{c}
\bar{\varphi} \\
& \mathcal{Q} \\
& \mathcal{Q} \\
\mathcal{Q} \\
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\begin{array}{c}
\mathcal{Q} \\
& \bar{\varphi} \\
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\mathcal{Q} \\
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\]

C. Morningstar Excited States 13
More complicated correlators

- two-meson to two-meson correlators (non isoscalar mesons)
Quantum numbers in toroidal box

- periodic boundary conditions in cubic box
  - not all directions equivalent ⇒ using $J^{PC}$ is wrong!!

- label stationary states of QCD in a periodic box using irreps of cubic space group even in continuum limit
  - zero momentum states: little group $O_h$
    $A_{1a}, A_{2g}, E_a, T_{1a}, T_{2a}, G_{1a}, G_{2a}, H_a, \quad a = g, u$
  - on-axis momenta: little group $C_{4v}$
    $A_1, A_2, B_1, B_2, E, \quad G_1, G_2$
  - planar-diagonal momenta: little group $C_{2v}$
    $A_1, A_2, B_1, B_2, \quad G_1, G_2$
  - cubic-diagonal momenta: little group $C_{3v}$
    $A_1, A_2, E, \quad F_1, F_2, G$

- include $G$ parity in some meson sectors (superscript $+$ or $-$)
Spin content of cubic box irreps

- numbers of occurrences of $\Lambda$ irreps in $J$ subduced

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Ensembles and run parameters

- focusing on two Monte Carlo ensembles
  - $(32^3|240)$: 412 configs $32^3 \times 256$, $m_\pi \approx 240$ MeV, $m_\pi L \sim 4.4$
  - $(24^3|390)$: 551 configs $24^3 \times 128$, $m_\pi \approx 390$ MeV, $m_\pi L \sim 5.7$

- anisotropic improved gluon action, clover quarks (stout links)
- QCD coupling $\beta = 1.5$ such that $a_s \sim 0.12$ fm, $a_t \sim 0.035$ fm
- strange quark mass $m_s = -0.0743$ nearly physical (using kaon)
- work in $m_u = m_d$ limit so $SU(2)$ isospin exact
- generated using RHMC, configs separated by 20 trajectories

- stout-link smearing in operators $\xi = 0.10$ and $n_\xi = 10$
- LapH smearing cutoff $\sigma_s^2 = 0.33$ such that
  - $N_v = 112$ for $24^3$ lattices
  - $N_v = 264$ for $32^3$ lattices

- source times:
  - 4 widely-separated $t_0$ values on $24^3$
  - 8 $t_0$ values used on $32^3$ lattice
$I = 1, \ S = 0, \ T_{1u}^+$ channel

- effective energies $\tilde{m}_{\text{eff}}(t)$ for levels 0 to 24 ($32^3|240$)
- energies obtained from two-exponential fits (B. Fahy, PhD thesis)
effective energies $\tilde{m}^{\text{eff}}(t)$ for levels 25 to 49
energies obtained from two-exponential fits
Level identification

- overlaps for various operators

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Excited States
Staircase of energy levels

- stationary state energies $I = 1$, $S = 0$, $T_{1u}^+$ channel on $(32^3 \times 256)$ anisotropic lattice

![Graph showing levels and mixing]

- $m/m_K$: Levels

Legend:
- Blue: single-hadron dominated
- Cyan: two-hadron dominated
- Green: significant mixing

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Excited States
Summary and comparison with experiment

- right: energies of $\bar{q}q$-dominant states as ratios over $m_K$ for $(32^3|240)$ ensemble
- left: experiment (masses and widths)

![Graph showing comparison between experiment and lattice results for excited states.](graph.png)
Issues

- infinite-volume resonance parameters from finite-volume energies
  - Luscher method too cumbersome, restrictive in applicability
  - use of hadron effective Hamiltonian techniques
- address presence of 3 and 4 meson states
- in other channels, address scalar particles in spectrum
  - scalar probe states need vacuum subtractions
  - hopefully can neglect due to OZI suppression
Bosonic $I = 1, S = 0, A_{1u}^-$ channel

- finite-volume stationary-state energies: “staircase” plot
- $32^3 \times 256$ lattice for $m_\pi \sim 240$ MeV
- use of single- and two-meson operators only
- blue: levels of max overlaps with SH optimized operators
Bosonic $I = 1$, $S = 0$, $E^{+}_{u}$ channel

- finite-volume stationary-state energies: “staircase” plot
- $32^3 \times 256$ lattice for $m_{\pi} \sim 240$ MeV
- use of single- and two-meson operators only
- blue: levels of max overlaps with SH optimized operators
Bosonic $I = 1$, $S = 0$, $T_{1g}^-$ channel

- finite-volume stationary-state energies: “staircase” plot
- $32^3 \times 256$ lattice for $m_\pi \sim 240$ MeV
- use of single- and two-meson operators only
- blue: levels of max overlaps with SH optimized operators

T1gm 1
Bosonic $I = 1, \ S = 0, \ T_{1u}^-$ channel

- finite-volume stationary-state energies: “staircase” plot
- $32^3 \times 256$ lattice for $m_\pi \sim 240$ MeV
- use of single- and two-meson operators only
- blue: levels of max overlaps with SH optimized operators
Bosonic $I = \frac{1}{2}, \quad S = 1, \quad T_{1u}$ channel

- kaon channel: effective energies $\tilde{m}^{\text{eff}}(t)$ for levels 0 to 8
- results for $32^3 \times 256$ lattice for $m_\pi \sim 240$ MeV
- two-exponential fits (Y.C. Jhang, PhD thesis)
Bosonic $I = \frac{1}{2}, \ S = 1, \ T_{1u}$ channel

- effective energies $\tilde{m}^{\text{eff}}(t)$ for levels 9 to 17
- results for $32^3 \times 256$ lattice for $m_\pi \sim 240$ MeV
- two-exponential fits
Bosonic $I = \frac{1}{2}$, $S = 1$, $T_{1u}$ channel

- effective energies $\tilde{m}^{\text{eff}}(t)$ for levels 18 to 23
- dashed lines show energies from single exponential fits
Bosonic $I = \frac{1}{2}, \ S = 1, \ T_{1u}$ channel

- finite-volume stationary-state energies: “staircase” plot
- $32^3 \times 256$ lattice for $m_\pi \sim 240$ MeV
- use of single- and two-meson operators only
- blue: levels of max overlaps with SH optimized operators

kaon T1u 32
Scattering phase shifts in lattice QCD timeline

- DeWitt 1956: finite-volume energies related to scattering phase shifts
- Lüscher 1984: quantum mechanics in a box
- Rummukainen and Gottlieb 1995: nonzero total momenta
- Kim, Sachrajda, and Sharpe 2005: field theoretic derivation
- explosion of papers since then
- generalized to arbitrary spin, multiple channels
Scattering phase shifts from finite-volume energies

- Correlator of two-particle operator $\sigma$ in finite volume

$$C^L(P) = \sigma \sigma^\dagger + \sigma iK \sigma^\dagger$$

+ $$\sigma iK iK \sigma^\dagger + \ldots$$

- Bethe-Salpeter kernel

$$iK = \times + \bigcirc + \bigcirc \bigcirc$$

+ $$\bigcirc + \bigcirc$$

- $C^\infty(P)$ has branch cuts where two-particle thresholds begin
- Momentum quantization in finite volume: cuts $\rightarrow$ series of poles
- $C^L$ poles: two-particle energy spectrum of finite volume theory
Phase shift from finite-volume energies (con’t)

- finite-volume momentum sum is infinite-volume integral plus correction $\mathcal{F}$

- define the following quantities: $A, A'$, invariant scattering amplitude $i\mathcal{M}$

\[
\begin{align*}
    A &= \sigma + \sigma iK \\
    &\quad + \sigma iK iK + \ldots \\
    A' &= \sigma^\dagger + iK \sigma^\dagger \\
    &\quad + iK iK \sigma^\dagger + \ldots \\
    i\mathcal{M} &= iK + iK iK + \ldots \\
    &\quad + iK iK iK + \ldots
\end{align*}
\]
Phase shifts from finite-volume energies (con’t)

- subtracted correlator \( C_{\text{sub}}(P) = C^L(P) - C^\infty(P) \) given by

\[
C_{\text{sub}}(P) = A \left( A' \right) + A \left( iM \right) A' + \ldots
\]

- sum geometric series

\[
C_{\text{sub}}(P) = A \mathcal{F} (1 - iM\mathcal{F})^{-1} A'.
\]

- poles of \( C_{\text{sub}}(P) \) are poles of \( C^L(P) \) from \( \det(1 - iM\mathcal{F}) = 0 \)

- key tool: for \( g_c(p) \) spatially contained and regular

\[
\frac{1}{L^3} \sum_p g_c(p) = \int \frac{d^3k}{(2\pi)^3} g_c(k) + O(e^{-mL})
\]

\[
\frac{1}{L^3} \sum_p \frac{g_c(p^2)}{(p^2 - a^2)} = \frac{1}{L^3} \sum_p \frac{g_c(a^2)}{(p^2 - a^2)} + \int \frac{d^3k}{(2\pi)^3} \frac{g_c(p^2) - g(a^2)}{(p^2 - a^2)} + O(e^{-mL})
\]
Phase shifts from finite-volume energies (con’t)

- work in spatial $L^3$ volume with periodic b.c.
- total momentum $\mathbf{P} = (2\pi/L)\mathbf{d}$, where $\mathbf{d}$ vector of integers
- masses $m_1$ and $m_2$ of particle 1 and 2
- calculate lab-frame energy $E$ of two-particle interacting state in lattice QCD
- boost to center-of-mass frame by defining:

$$E_{\text{cm}} = \sqrt{E^2 - \mathbf{P}^2}, \quad \gamma = \frac{E}{E_{\text{cm}}},$$

$$q_{\text{cm}}^2 = \frac{1}{4}E_{\text{cm}}^2 - \frac{1}{2}(m_1^2 + m_2^2) + \frac{(m_1^2 - m_2^2)^2}{4E_{\text{cm}}^2},$$

$$u^2 = \frac{L^2 q_{\text{cm}}^2}{(2\pi)^2}, \quad s = \left(1 + \frac{(m_1^2 - m_2^2)}{E_{\text{cm}}^2}\right)d$$

- $E$ related to $S$ matrix (and phase shifts) by

$$\det[1 + F(s,\gamma,u)(S - 1)] = 0,$$

where $F$ matrix defined next slide
Phase shifts from finite-volume energies (con’t)

- $F$ matrix in $JLS$ basis states given by

$$F_{J'm_j,L'S'a'}^{(s,\gamma,u)}; Jm_jLSa = \frac{\rho_a}{2} \delta_{a'a} \delta_{S'S} \left\{ \delta_{J'J} \delta_{m_j,m_j} \delta_{L'L} ight\}$$

$$+ W_{L'm_{L'}}; Lm_L \langle J'm_J|L'm_{L'}, Sm_S \rangle \langle Lm_L, Sm_S | Jm_J \rangle \right\},$$

- total angular mom $J, J'$, orbital mom $L, L'$, intrinsic spin $S, S'$
- $a, a'$ channel labels
- $\rho_a = 1$ distinguishable particles, $\rho_a = \frac{1}{2}$ identical

$$W_{L'm_{L'}; Lm_L}^{(s,\gamma,u)} = \frac{2i}{\pi^2 u^{l+1}} \mathcal{Z}_{lm}(s, \gamma, u^2) \int d^2\Omega \ Y_{L'm_{L'}}^\ast(\Omega) Y_{lm}(\Omega) Y_{Lm_L}(\Omega)$$

- Rummukainen-Gottlieb-Lüscher (RGL) shifted zeta functions $\mathcal{Z}_{lm}$ defined next slide
- $F^{(s,\gamma,u)}$ diagonal in channel space, mixes different $J, J'$
- recall $S$ diagonal in angular momentum, but off-diagonal in channel space
compute $Z_{lm}$ using

$$Z_{lm}(s, \gamma, u^2) = \sum_{n \in \mathbb{Z}^3} \frac{\mathcal{Y}_{lm}(z)}{(z^2 - u^2)} e^{-\Lambda(z^2 - u^2)} + \delta_{l0} \frac{\gamma \pi}{\sqrt{\Lambda}} F_0(\Lambda u^2)$$

$$+ \frac{i^l \gamma}{\Lambda^{l+1/2}} \int_0^1 dt \left( \frac{\pi}{t} \right)^{l+3/2} e^{\Lambda tu^2} \sum_{n \in \mathbb{Z}^3 \atop n \neq 0} e^{\pi in \cdot s} \mathcal{Y}_{lm}(w) e^{-\pi^2 w^2 / (t \Lambda)}$$

where

$$z = n - \gamma^{-1} \left[ \frac{1}{2} + (\gamma - 1)s^{-2} n \cdot s \right] s,$$

$$w = n - (1 - \gamma)s^{-2} s \cdot ns,$$

$$\mathcal{Y}_{lm}(x) = |x|^l Y_{lm}(\hat{x})$$

$$F_0(x) = -1 + \frac{1}{2} \int_0^1 dt \frac{e^{tx} - 1}{t^{3/2}}$$

choose $\Lambda \approx 1$ for convergence of the summation

integral done Gauss-Legendre quadrature

$F_0(x)$ given in terms of Dawson or erf function
Block diagonalization of $F$ matrix

- quantization condition is large determinant relation:
  \[
  \det[1 + F^{(s,\gamma,u)}(S - 1)] = 0
  \]
- define the matrix
  \[
  B_{J'm',L'S'a'}^{(R)}; \ Jm_jLSa = \delta_{J'J} \delta_{L'L} \delta_{S'S} \delta_{a'a} D_{m'm_j}^{(J)}(R)
  \]
- can show that under lattice symmetry operator $R$,
  \[
  F^{(Rs,\gamma,u)} = B^{(R)} F^{(s,\gamma,u)} B^{(R)^\dagger}
  \]
- can block diagonalize $F$ by diagonalizing $D_{m'm_j}^{(J)}(R)$ for each $J$
- change of basis: little group irrep $\Lambda$, row $\lambda$, $n$ occurrence of $\Lambda$ in $D_{m'm_j}^{(J)}(R)$
  \[
  |\Lambda\lambda nJLSa\rangle = \sum_m c_{Jm_j}^{\Lambda\lambda n} |Jm_jLSa\rangle
  \]
- $F$ diagonal in $\Lambda$, $\lambda$, but not in $n_{\Lambda m_j}$
- can now focus on the matrix elements:
  \[
  F^{(s,\gamma,u)}(\Lambda,\lambda)_{J'n'L'S'a'; JnLSa}
  \]
for \(P\)-wave phase shift \(\delta_1(E_{\text{cm}})\) for \(\pi\pi\ I = 1\) scattering

define

\[
w_{lm} = \frac{Z_{lm}(s, \gamma, u^2)}{\gamma \pi^{3/2} u^{l+1}}
\]

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<td>(\text{Re } w_{0,0})</td>
</tr>
<tr>
<td>(0,0,1)</td>
<td>(A_1^+)</td>
<td>(\text{Re } w_{0,0} + \frac{2}{\sqrt{5}} \text{Re } w_{2,0})</td>
</tr>
<tr>
<td></td>
<td>(E^+)</td>
<td>(\text{Re } w_{0,0} - \frac{1}{\sqrt{5}} \text{Re } w_{2,0})</td>
</tr>
<tr>
<td>(0,1,1)</td>
<td>(A_1^+)</td>
<td>(\text{Re } w_{0,0} + \frac{1}{2\sqrt{5}} \text{Re } w_{2,0} - \sqrt{\frac{6}{5}} \text{Im } w_{2,1} - \sqrt{\frac{3}{10}} \text{Re } w_{2,2})</td>
</tr>
<tr>
<td></td>
<td>(B_1^+)</td>
<td>(\text{Re } w_{0,0} - \frac{1}{\sqrt{5}} \text{Re } w_{2,0} + \sqrt{\frac{6}{5}} \text{Re } w_{2,2})</td>
</tr>
<tr>
<td></td>
<td>(B_2^+)</td>
<td>(\text{Re } w_{0,0} + \frac{1}{2\sqrt{5}} \text{Re } w_{2,0} + \sqrt{\frac{6}{5}} \text{Im } w_{2,1} - \sqrt{\frac{3}{10}} \text{Re } w_{2,2})</td>
</tr>
<tr>
<td>(1,1,1)</td>
<td>(A_1^+)</td>
<td>(\text{Re } w_{0,0} + 2 \sqrt{\frac{6}{5}} \text{Im } w_{2,2})</td>
</tr>
<tr>
<td></td>
<td>(E^+)</td>
<td>(\text{Re } w_{0,0} - \sqrt{\frac{6}{5}} \text{Im } w_{2,2})</td>
</tr>
</tbody>
</table>
Finite-volume $\pi\pi$ $I = 1$ energies

- $\pi\pi$-state energies for various $d^2$

\[
E_{+1u} = 0, \ T_{2d+1} = 1, \ A_{2d+1} = 1, \ E_{2d+1} = 2, \ A_{2d+1} = 2, \ B_{2d+2} = 2, \ B_{2d+2} = 2, \ ...
\]
$I = 1\ \pi\pi$ scattering phase shift and width of the $\rho$

- results $32^3 \times 256$, $m_\pi \approx 240$ MeV:
  - $g_{\rho\pi\pi} = 5.99(26)$, $m_\rho/m_\pi = 3.350(24)$, $\chi^2$/dof = 1.04

- fit
  - $g_{\rho\pi\pi}^2 q_{cm}^3 \cot(\delta_1) = 6\pi E_{cm}(m_\rho^2 - E_{cm}^2)$
$I = 1 \pi \pi$ scattering phase shift and width of the $\rho$

C. Morningstar

Excited States
$I = 1 \, \pi\pi$ scattering phase shift and width of the $\rho$

- compendium of results for $g_{\rho\pi\pi}$

![Graph showing $m_\rho/m_\pi$ and $g_{\rho\pi\pi}$ as functions of $m_\pi^2$ and $m_\pi$.]
Effective Hamiltonian method

- relating finite-volume energies to resonance parameters via “Lüscher method” very complicated
- alternative: use an effective hadron Hamiltonian matrix
  - Wu et al, PRC 90, 055206 (2014)
- use single and two-particle states as basis states
- interaction terms from symmetry + assumed form wrt momenta
- parameters of Hamiltonian determined from fits to finite-volume spectra
- Lippmann-Schwinger (or other methods) to extract infinite-volume resonances
Tetraquark operators

- determine impact on spectrum when tetraquark operators included
- single-site and displaced quarks


J. Bulava et al., *$I = 1$ and $I = 2$ $\pi\pi$ scattering phase shifts from $N_f = 2 + 1$ lattice QCD*, submitted to Nucl. Phys. B. (2016).
Conclusions

- Excited states are difficult!
- Crucial role of interpolating operators for excited-state studies in lattice QCD
- Large number of finite-volume energies in large number of channels now possible
- Stochastic LapH method works very well
  - Allows evaluation of all needed quark-line diagrams
- Scattering phase shifts can be computed
- Infinite-volume resonance parameters from finite-volume energies → Need for simpler effective Hamiltonian/field theory techniques
- Role of tetraquark operators to be studied