Transport Coefficients and Lattice QCD

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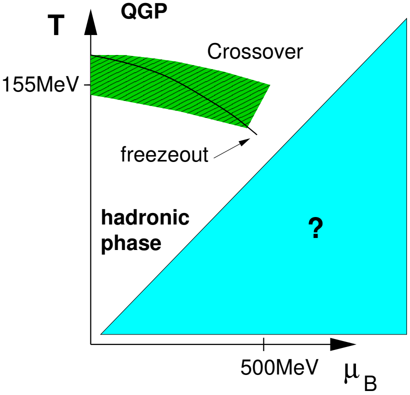
Outline

- **Brief overview of equilibrium properties**

- **Near-equilibrium properties**
  - formalism
  - two channels: light-quark pseudoscalar and vector channels
  - variational method for dense spectrum?
  - screening masses and their relation to transport properties.



Motivation

Strongly interacting matter at temperatures $T = 100 - 500\,\text{MeV}$

- probed in heavy-ion collisions
- state of matter for the first microsecond after Big Bang

Thermal physics:

$$\langle A \rangle = \frac{1}{Z} \mathrm{Tr}\,\{e^{-\beta H} A\}, \qquad Z = \mathrm{Tr}\,\{e^{-\beta H}\}$$

Matsubara formalism particularly well-suited for **equilibrium physics:**
path integral formulation

- imaginary time direction has an extent $\hbar/(k_B T)$
- boson fields have periodic, fermion fields antiperiodic boundary conditions.

$\longrightarrow$ particularly well suited for lattice QCD: $Z = \int DU\, D\bar{\psi}\, D\psi\, e^{-S}$.

Thermodynamic potentials

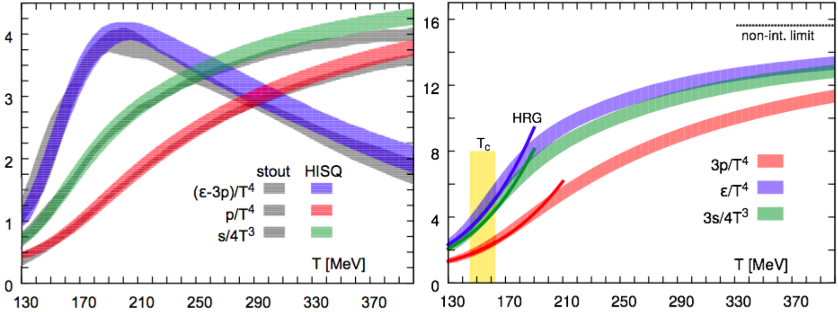


Fig. from review by Soltz et al. 1502.02296

- at $T = 260\text{MeV}$, $p_{\text{norm}} \equiv p/p_{\text{SB}} \approx 1/2$:
  far from weakly interacting quarks and gluons.
- $(e - 3p)/[\frac{3}{4}(e+p)] \approx 1/3$: large departure from a scale-invariant system.
- HRG model works well up to $T = 160\,\text{MeV}$.

Near-equilibrium properties

Typical questions:

- What quasiparticles are there in the system?
- How fast does a perturbation of a given wavelength dissipate in the system?
- What is the production rate of photons?

Formalism

- Relation between the Euclidean correlator and the spectral function:

$$G_E(x_0,\mathbf{p}) = \int d^3x\; e^{-i\mathbf{p}\cdot\mathbf{x}}\,\langle J(x)J(0)\rangle \overset{\star}{=} \int_0^\infty \frac{d\omega}{2\pi}\;\rho(\omega,\mathbf{p})\,\frac{\cosh[\omega(\beta/2-x_0)]}{\sinh[\omega\beta/2]}.$$

Alternatively,

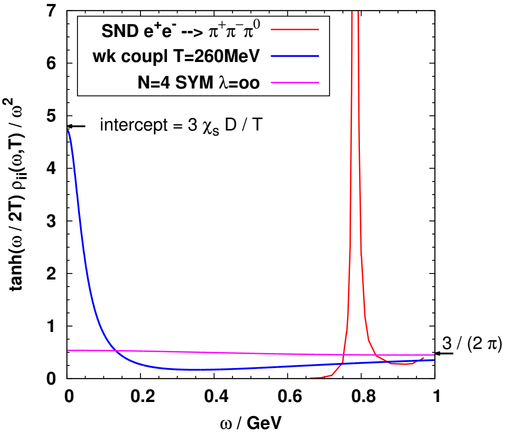
$$G_E(\omega_n,\boldsymbol{x}) = \int_0^\beta dx_0\; e^{i\omega_n x_0}\;\langle J(x)J(0)\rangle \overset{\star}{=} 2\int_0^\infty d\omega\;\frac{\omega\;\rho(\omega,\boldsymbol{x})}{\omega^2+\omega_n^2}.$$

- for $J = J_i^{\rm em}$ electromagnetic current, $\rho(\omega,\mathbf{0}) \overset{\omega\to 0}{\sim} 6\chi_s D\omega$
  ($\chi_s =$ static susceptibility of electric charge, $D=$ diffusion coefficient)
- in the low-T phase, $J_i^{\rm em}$ can excite e.g. an ω-meson-like quasiparticle.
- photon rate: $\frac{d\Gamma}{d^3k} = \frac{e^2\sum_f Q_f^2}{2(2\pi)^3\,k}\,\frac{\rho(k,\boldsymbol{k})}{e^{\beta k}-1}$

$\star$ inverse problem for $\rho(\omega,\boldsymbol{p})$

Motivation: expected thermal changes in spectral functions

Isoscalar vector channel: spectral fct. of $J_i = \frac{1}{\sqrt{2}}(\bar{u}\gamma_i \bar{u} + \bar{d}\gamma_i d)$



- presence of weakly coupled quasiparticles ⇒ transport peak at $\omega = 0$; is it really there at $T \approx 260$MeV ?

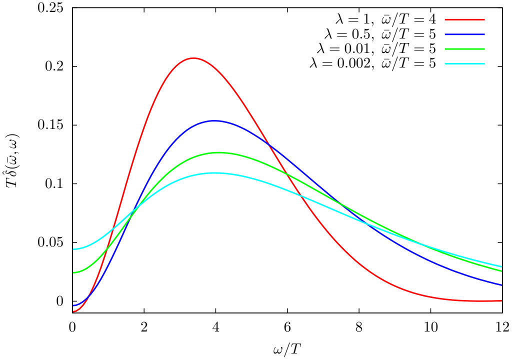
SND hep-ex/0305049

$D =$ diffusion coefficient; $\chi_s =$ static susceptibility.

Some basics on the inverse problem

$$\text{Linearity:} \qquad \sum_{i=1}^{n} q_i(\bar\omega)\, G(t_i) = \int_0^\infty \frac{d\omega}{2\pi}\, \rho(\omega)\, \underbrace{\sum_{i=1}^{n} q_i(\bar\omega)\, \frac{\cosh[\omega(\beta/2 - t_i)]}{\sinh[\omega\beta/2]}}_{\widehat{\delta}(\bar\omega,\omega)}$$

- choose the coefficients $q_i(\bar\omega)$ so that the 'resolution function' $\widehat{\delta}(\bar\omega, \omega)$ is as narrowly peaked around a given frequency $\bar\omega$ as possible (idea behind the Backus-Gilbert method, [used in Robaina et al. 1506.05732])



Resolution function at $\bar\omega = 4$ for $N_t = 24$, $t_i/a = 5, \ldots 12$.

The pion quasiparticle in the low-temperature phase

- Chiral symmetry is spontaneously broken for $T < T_c$: $-\langle\bar\psi\psi\rangle > 0$.
- Goldstone theorem $\Rightarrow$ a divergent spatial correlation length exists in the limit $m \to 0$.
- somewhat less obvious: a massless real-time excitation exists: the pion quasiparticle.
- dispersion relation: $\omega_{\boldsymbol{p}} = u\sqrt{m_\pi^2 + \boldsymbol{p}^2} + \ldots$; m_π = screening mass(!)
  [Son and Stephanov, PRD 66, 076011 (2002)]
- pion dominates Euclidean two-point function of A_0 and of P at $x_0 = \beta/2$

$$T = 0: \qquad \text{pion mass} = 267(2)\,\text{MeV}$$

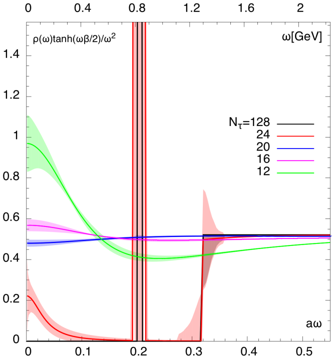
$$\swarrow \qquad \searrow$$

$$T = 169\text{MeV}: \qquad \text{quasiparticle mass} = 223(4)\text{MeV} \qquad \text{screening mass} = 303(4)\text{MeV}.$$

Implications for the hadron resonance gas model!?

Robaina et al. 1406.5602; 1506.05732

An all-temperature analysis of the isovector vector channel at $p = 0$



- global fit with to all temperatures using sum rule $\int_0^\infty d\omega\, \Delta\rho(\omega)/\omega = 0$ and $\rho(\omega) \sim A\omega^2$ at large ω, A temperature-independent (OPE).
- area under transport peak $\sim \chi_s \langle v^2 \rangle \rightsquigarrow$ sensitive to pion dispersion relation for $T < T_c$.
- gradual disappearance of the ρ as T increases.

Francis et al. ($N_f = 2$), 1512.07249, $N_t \times 64^3$ lattices, $m_\pi|_{T=0} = 270\mathrm{MeV}$

Screening masses: static and non-static

Consider perturbating the Hamiltonian,

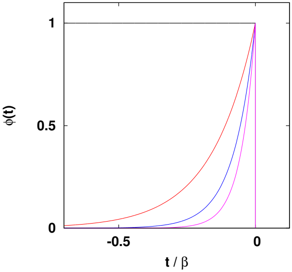
$$\hat{H}_\phi(t) = \hat{H} - \int d^3y\ \phi(t,\boldsymbol{y})\hat{J}(t,\boldsymbol{y}),$$

with the external perturbation given by

$$\phi(t,\boldsymbol{y}) = \delta(\boldsymbol{y})e^{\omega t}\theta(-t), \qquad \omega \geq 0.$$

Linear response $\Rightarrow$

$$\delta\langle J(t=0,\boldsymbol{x})\rangle = \underbrace{G_E^{JJ}(\omega_n,\boldsymbol{x})}_{\text{Euclidean corr.}}, \qquad \text{for } \omega = \omega_n = 2\pi T n.$$



Correlation length in Matsubara sector ω_n = length scale over which a perturbation with the time dependence $e^{\omega_n t}$ is screened ($n \geq 0$).

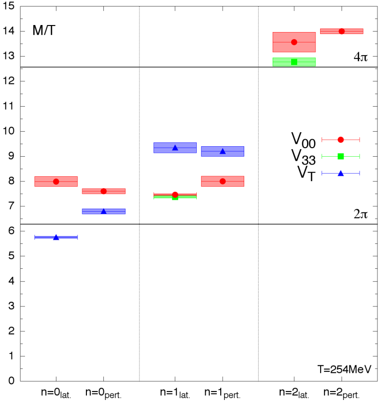
Screening masses at high temperatures

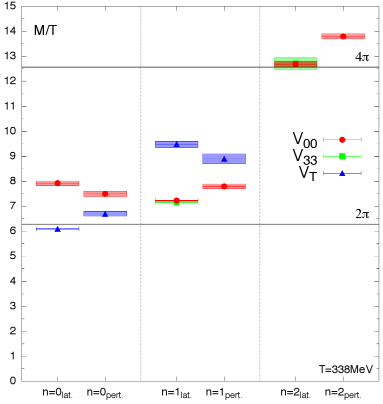
Weak-coupling picture of flavor-non-singlet screening masses:

- fermions have an effective mass of at least $\pi T \Rightarrow$ dimensional reduction
- they form non-relativistic, 2+1d bound states of size O(m_E^{-1})
  Laine, Vepsalainen hep-ph/0311268
- expect bound state to be described by a Schrödinger equation in 2+1d.
- Non-static sector: potential has a connection with an effective potential used in the calculation of the dilepton production rate

  [Aurenche, Gelis, Moore, Zakaret hep-ph/0211036; Caron-Huot 0811.1603; Panero, Rummukainen, Schäfer 1307.5850].

Vector screening masses: lattice vs. EFT



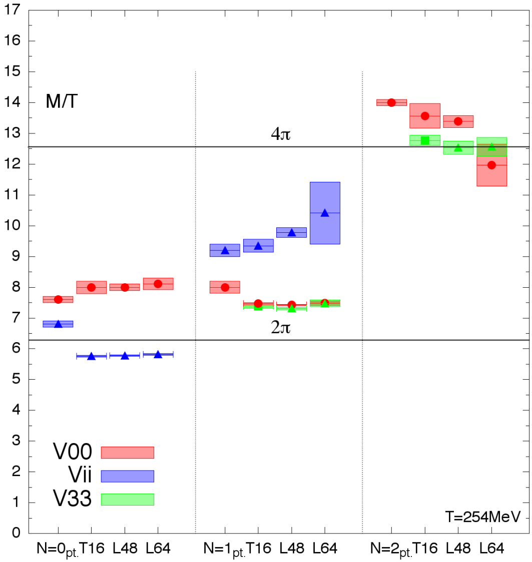


$T = 254$ MeV $\qquad$ $T = 340$ MeV

Satisfactory agreement between lattice QCD and the EFT predictions.

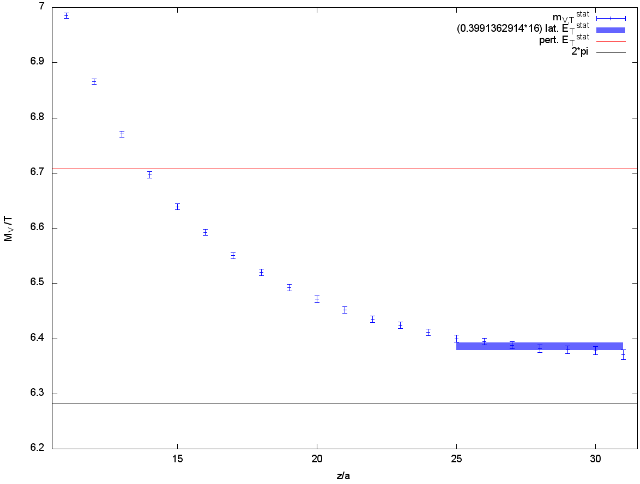
Brandt et al. 1404.2404; $N_t = 16$ and $N_t = 12$, $N_s = 64$; $m_\pi|_T = 0 = 270\text{MeV}$

Checking for systematics at $T = 254$MeV



EFT vs. 16×64^3 vs. 12×48^3 vs. 12×64^3 Francis et al. Preliminary

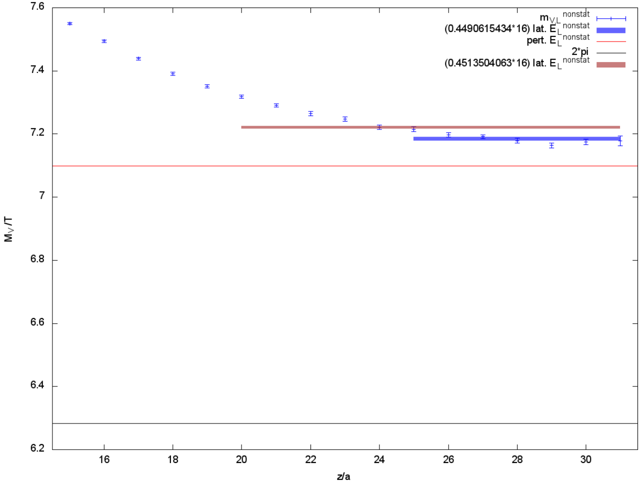
Static transverse vector screening mass at $T = 508$MeV



- now screening mass above $2\pi T$!
  (red line is $\mathrm{O}(g^2)$ prediction; black line is $2\pi T$).

[A. Steinberg, K. Zapp et al., in prep.; 16×64^3]

$n = 1$ longitudinal vector screening mass at $T = 508$MeV



- 1%-level agreement with $\mathrm{O}(g^2)$ prediction (red line; black line is $2\pi T$).

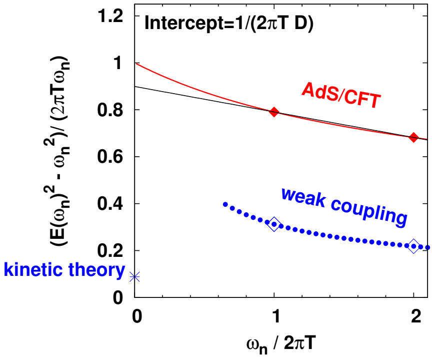
[A. Steinberg, K. Zapp et al., in prep.]

Non-static screening masses and transport coefficients

Linear response along with a constitutive equation for the vector current $\boldsymbol{J}$ $\Rightarrow$

$$G_E^{J_0 J_0}(\omega_n, k) \overset{\omega_n, k\to 0}{=\!=} \frac{\chi_s D k^2}{\omega_n + D k^2} \qquad \Rightarrow \qquad E(\omega_n)^2 \overset{\omega_n\to 0}{\sim} \frac{\omega_n}{D}.$$

χ_s = static susceptibility, D = diffusion coefficient, $E(\omega_n)$ = screening mass in sector ω_n



In the limit $T \to \infty$, extrapolating the screening masses in the lowest Matsubara sectors to $\omega_n = 0$ gives the correct result, $1/(T\,D) = 0$.

Brandt, Francis, Laine, HM 1408.5917; Kinetic theory: Arnold, Moore & Yaffe hep-ph/0111107

Diffusion Coeff. from analytic continuation of screening correlator

$$G_E(\omega_n, \boldsymbol{k}_\perp = 0, z) = 2\int_0^\infty d\omega\; \frac{\omega\;\rho(\omega, z)}{\omega^2 + \omega_n^2}$$

- this spectral representation provides the analytic continuation of G_E
- for large z: given $G_E(\omega_n, \boldsymbol{k}_\perp = 0, z)$, $n = 0, 1, 2, \dots$ reconstruct

$$G_E(\omega_E, \boldsymbol{k}_\perp = 0, z) = 2\int_0^\infty d\omega\; \frac{\omega\;\rho(\omega, z)}{\omega^2 + \omega_E^2}.$$

- fit $G_E(\omega_E, \boldsymbol{k}_\perp = 0, z) \sim e^{-E(\omega_E)|z|}$ to get $E(\omega_E)$.
- observe diffusive regime $E(\omega_E)^2 \overset{\omega_E \to 0}{\sim} \frac{\omega_E}{D}$?
- NB. causality $\Rightarrow E(\omega_n) \geq |\omega_n|$, because Wightman correlator

$$G_>(t, \boldsymbol{x}) \equiv \frac{1}{Z}\sum_n e^{-\beta E_n}\langle n|j_0(t, \boldsymbol{x})j_0(0)|n\rangle = T\sum_n e^{\omega_n t} G_E(\omega_n, \boldsymbol{x})$$

should be analytic in the spacelike region $t^2 - \boldsymbol{x}^2 < 0$.

Variational method for dense spectra

- The variational method (using a basis of N operators with same quantum numbers) is very successful at $T = 0$ to determine the low-lying spectrum see e.g. Blossier et al., 0902.1265]. Can it in some sense be generalized to $T > 0$ and/or to the higher part of the spectrum?

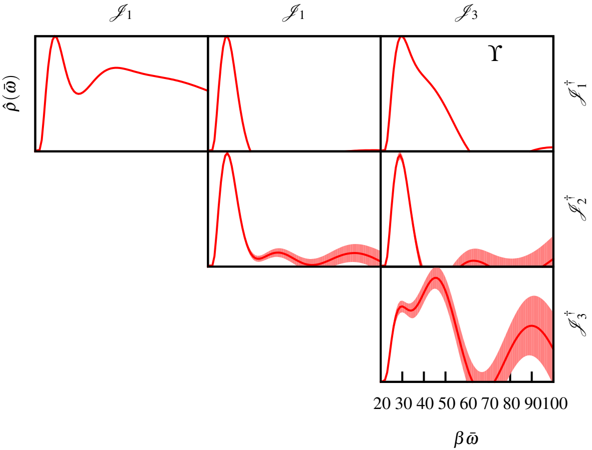
- At $T = 0$, large Euclidean time x_0 is used to effectively 'reduce' the Hilbert space to an N-dimensional subspace.

- For the higher-lying spectrum, this is no longer practical: the spectrum is too dense.
  And at finite-temperature, $\exists$ kinematic limitation $0 \leq x_0 < 1/(2T)$.

- $\rightsquigarrow$ make use of a matrix of Backus-Gilbert spectral functions, $\widehat{\rho}_{ij}(\bar{\omega})$.

T. Harris, HM, D. Robaina; T. Harris, talk at Trento workshop 2-6 May 2016

Υ channel



BG matrix estimator $\hat{\rho}_{ij}(\bar{\omega})$ in Υ channel for β/a=128

Operator basis $\{\mathcal{J}_i^\dagger(x) = \sum_k \chi_i^\dagger(x)\sigma_k\psi_i(x)\}$ where $\psi_1(x) \equiv \psi(x)$ and

$$\psi_2(y) \equiv \sum_{\boldsymbol{x}} e^{-(\boldsymbol{x}-\boldsymbol{y})^2/\sigma^2}\psi(x), \qquad \psi_3(y) \equiv \sum_{\boldsymbol{x}} (4\frac{(\boldsymbol{x}-\boldsymbol{y})^2}{\sigma^2} - 3)e^{-(\boldsymbol{x}-\boldsymbol{y})^2/\sigma^2}\psi(x).$$

Our proposal

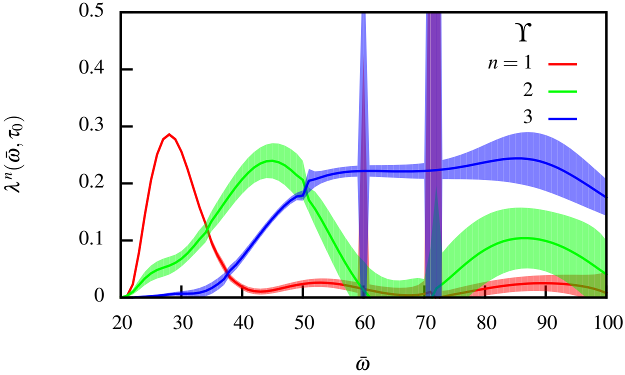
Solve the GEVP

$$\widehat{\rho}_{ij}(\bar{\omega})v_j^n(\bar{\omega}) = \lambda^n(\bar{\omega}, x_0)G_{E,ij}(t_o)v_j^n(\bar{\omega}),$$

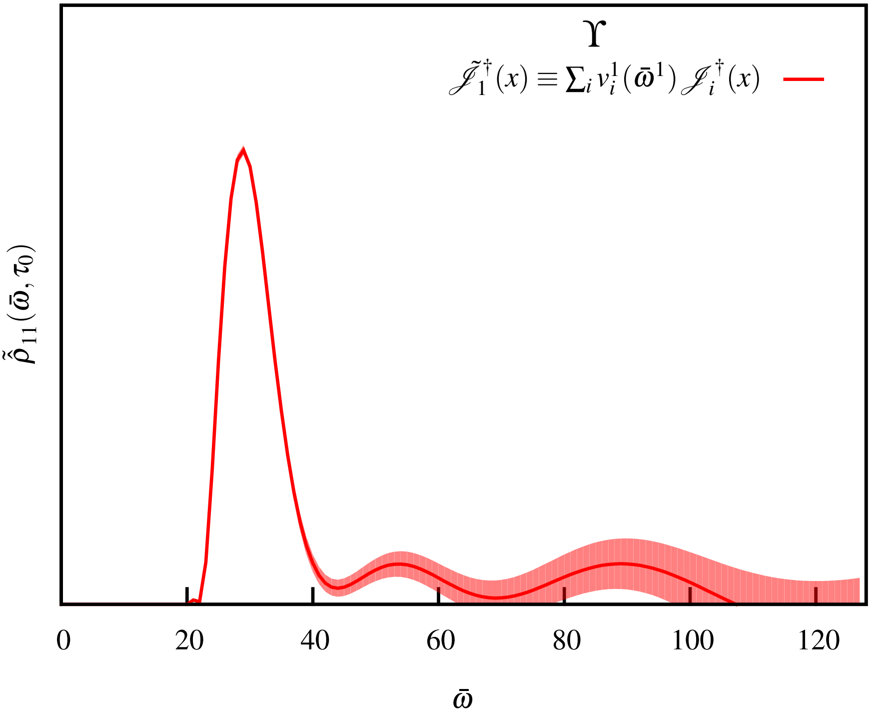
where $\widehat{\rho}_{ij}(\bar{\omega}) = \sum_\ell q_\ell(\bar{\omega})G_{E,ij}(t_\ell)$ is the Backus-Gilbert spectral function.

- Corresponds to extremizing $\Phi(v) = (v, \widehat{\rho}(\bar{\omega})\ v) + \lambda(v, G_E(t_o)\ v)$ "maximize the local spectral weight of the operator for a fixed normalization in the UV";
  the width is given by the width of the resolution function $\widehat{\delta}(\bar{\omega}, \omega)$.
- If the local spectrum around $\bar{\omega}$ contains r states, $\mathrm{rank}(\widehat{\rho}_{ij}(\bar{\omega})) = r$, because residue of pole contribution factorizes, $G_{E,ij}(t) \sim O_i^n\ O_j^n e^{-E_n t}$; diagnostic to detect resonances/quasiparticles.
- If $O_v(\bar{\omega}) = \sum_{j=1}^N v_j^n(\bar{\omega})O_j$ couples best to region around $\bar{\omega}$, use $\langle O_v^n V_\mu \rangle$ to measure coupling of the e.m. current to that region.

Υ channel



Eigenvalues in Υ channel for β/a=128



Υ
$\tilde{\mathscr{J}}_1^\dagger(x) \equiv \sum_i v_i^1(\bar{\omega}^1)\,\mathscr{J}_i^\dagger(x)$
$\tilde{\hat{\rho}}_{11}(\bar{\omega}, \tau_0)$
0
20
40
60
80
100
120
$\bar{\omega}$

Conclusion

Progress in lattice QCD on near-equilibrium quantities:

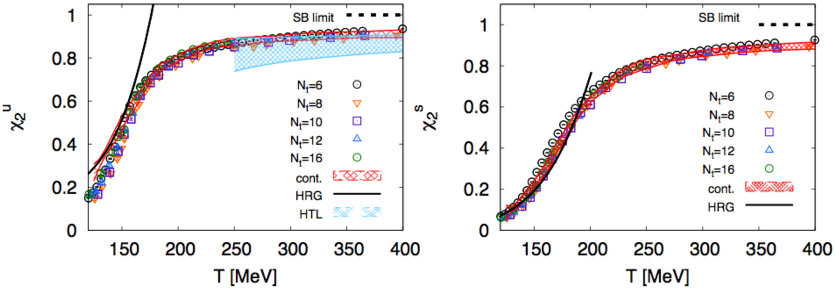
- current data quality: $N_t \approx 24$, few-permille precision on correlation functions, quenched continuum results.
- variational method can be useful also when individual energy eigenstates cannot be resolved;
  application at $T = 0$: determine R-ratio with moderate frequency resolution above limit of applicability of Lüscher's finite-volume formalism;
  NB. $\widehat{\rho}(\omega)$ has a smooth infinite-volume limit, $\rho(\omega)$ does not.
- screening masses & relevance to diffusion coefficient D and shear viscosity η.

Backup slides

4 topics

- the pion quasiparticle in the low-temperature phase of QCD
- spectral functions in the vector channel
- screening masses and their physical interpretation
- a variational method for dense spectrum.

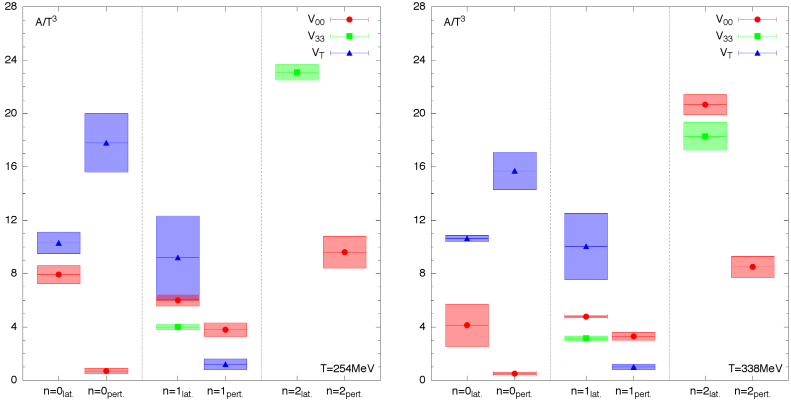
Deconfinement: does it coincide with chiral restoration?



- Not a completely sharp question.
- Light-quark number susceptibility: suggests that deconfinement occurs practically at the same temperature as chiral restoration.
- strangeness fluctuations: rise delayed by about $\Delta T = 20$MeV.
- Successful predictions of the hadron resonance gas model (HRG).

Fig. from S. Borsanyi et al. 1112.4416

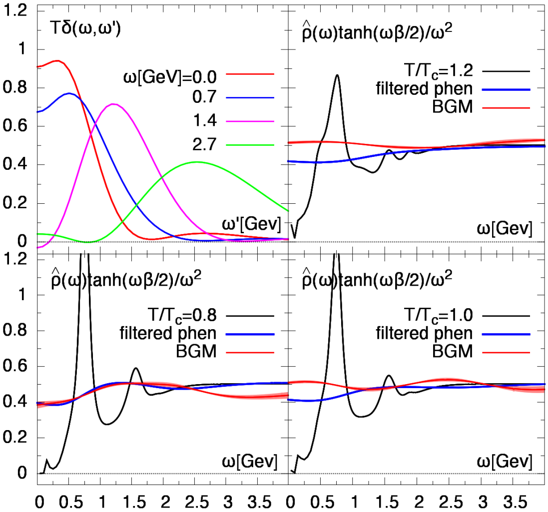
Amplitudes of vector screening states: lattice vs. EFT



$T = 250$ MeV $\qquad$ $T = 340$ MeV

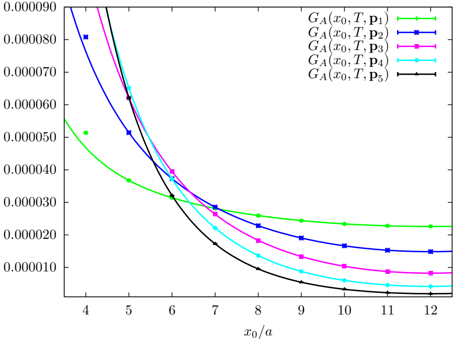
Prediction for the amplitude $\langle B|V_0|0\rangle$ is harder to get; better with non-pert. potential.

Comparison with phenomenological models



Francis, Brandt, Jäger, HM 1512.07249; model by Rapp & Hohler, Phys. Lett. B 731, 103 (2014).

Pion quasiparticle: test of the dispersion relation



- also the residue in two-point function of A_0 and of P are predicted
- dispersion relation & residue compatible with correlators at small $\mathbf{p} \neq 0$.

$$G_A(x_0, \mathbf{p}) = \frac{1}{3} \int d^3x \, e^{i\mathbf{p}\cdot\mathbf{x}} \, \langle A_0^a(x) A_0^a(0) \rangle = \int_0^\infty \frac{d\omega}{2\pi} \, \rho^A(\omega, \mathbf{p}) \frac{\cosh[\omega(\beta/2 - x_0)]}{\sinh[\omega\beta/2]}.$$

$$\text{Ansatz} : \rho^A(\omega, \mathbf{p}) = a_1(\mathbf{p})\delta(\omega - \omega_{\mathbf{p}}) + a_2(\mathbf{p})(1 - e^{-\omega\beta})\theta(\omega - c).$$

24×64^3 thermal ensemble, $T = 169$MeV, $m_\pi|_{T=0} = 270$MeV   1506.05732.

Portrait of QCD at finite temperature

From the lattice:

- low-T phase: hadron resonance gas model describes equilibrium properties very well
- chiral + deconfinement crossover transition around $T = 155\text{MeV}$
- high-T phase: multiplicity of degrees of freedom consistent with quarks+gluons
- ...but many quantities far from weak-coupling predictions at least until $T \approx 2.5T_c$.

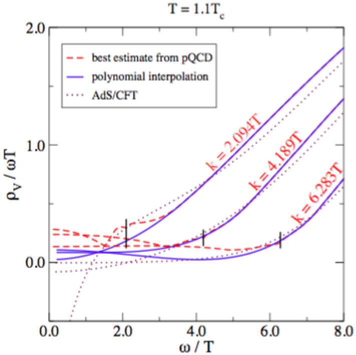
In addition, heavy-ion phenomenology points to a medium with very small shear viscosity/entropy density in the range $T_c \lesssim T \lesssim 2.5T_c$, e.g.

$$\eta/s \approx \begin{cases} 0.12 & \text{RHIC} \\ 0.2 & \text{ALICE} \end{cases}$$

Gale, Jeon, Schenke 1301.5893; White Paper 1502.02730

All this indicates that the partonic degrees of freedom are strongly correlated.

Additional information at non-vanishing spatial momentum



- allows for additional constraints on the spectral function
- impact on the diffusion coefficient D and the photon production rate (from $\omega = |\boldsymbol{k}|$)

Ghiglieri, Kaczmarek, Laine, F. Meyer 1604.07544; see also Foley et al. hep-lat/0610061, HM 0907.4095

Spectral sum rules for $\Delta\rho(\omega,\boldsymbol{k},T) \equiv \rho(\omega,\boldsymbol{k},T) - \rho(\omega,\boldsymbol{k},0)$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\,\omega\,\Delta\rho_V^L(\omega,\boldsymbol{k},T) = 0, \qquad \forall\boldsymbol{k} \qquad [1107.4388]$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\omega}{\omega}\Delta\rho_V^L(\omega,\boldsymbol{k},T) = \chi_s - \kappa_l\boldsymbol{k}^{\,2} + \mathrm{O}(|\boldsymbol{k}|^4),$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\omega}{\omega}\Delta\rho_V^T(\omega,\boldsymbol{k},T) = \kappa_t\boldsymbol{k}^{\,2} + \mathrm{O}(|\boldsymbol{k}|^4),$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\;\omega\;\Delta\rho_{\mathrm{A}}^L(\omega,\boldsymbol{k},T) = -m\langle\bar\psi\psi\rangle\Big|_0^T, \qquad \forall\boldsymbol{k} \qquad [1406.5602]$$

$$\vdots$$

$\exists$ interpretation of κ_l and κ_t in terms of screening/antiscreening of electric probe charges and currents placed in the medium   Brandt et al. 1310.5160

$$\frac{1}{3}\int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\,\langle V_0^a(x)V_0^a(0)\rangle = \int_0^\infty \frac{d\omega}{2\pi}\,\rho_V^L(\omega,\boldsymbol{k},T)\,\frac{\cosh\omega(\beta/2-x_0)}{\sinh\omega\beta/2},$$

$$-\tfrac{1}{6}\left(\delta_{il} - \frac{k_ik_l}{\boldsymbol{k}^2}\right)\int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\,\langle V_i^a(x)V_l^a(0)\rangle = \int_0^\infty \frac{d\omega}{2\pi}\,\rho_V^T(\omega,\boldsymbol{k},T)\,\frac{\cosh\omega(\beta/2-x_0)}{\sinh\omega\beta/2},$$

$$\frac{1}{3}\int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\,\langle A_0^a(0)A_0^a(x)\rangle = \int_0^\infty \frac{d\omega}{2\pi}\,\rho_{\mathrm{A}}^L(\omega,\boldsymbol{k},T)\,\frac{\cosh(\omega(\beta/2-x_0))}{\sinh(\omega\beta/2)}$$

Some basics on the inverse problem

$$\text{Linearity:} \quad \sum_{i=1}^{n} c_i(\bar\omega)\, G(t_i) = \int_0^\infty \frac{d\omega}{2\pi}\, \rho(\omega)\, \underbrace{\sum_{i=1}^{n} c_i(\bar\omega)\, \frac{\cosh[\omega(\beta/2 - t_i)]}{\sinh[\omega\beta/2]}}_{\widehat{\delta}(\bar\omega,\omega)}$$

- For given $\{t_i\}$, a certain resolution in frequency can be achieved; however, the required c_i are strongly oscillating (ill-posed problem)
- $\Rightarrow$ finite accuracy of data further limits the resolution
- if you *know* a priori that the spectral function is slowly varying on the scale $\Delta\omega \sim T$ the problem is again well posed.
- problem: whether there is a narrow transport peak or narrow quasiparticle peaks is precisely what we want to know.

Methods used: fit ansatz; maximum entropy method (MEM); new Bayesian method [Burnier & Rothkopf 1307.6106], S. Kim et al. 1511.04151; stochastic optimization method, H.-T. Shu et al. (1510.02901) and 'stochastic analytic inference' (H. Ohno et al.).